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Congruences modulo powers of 2 for 2 and 3-regular cubic partition pair

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Abstract
Let $B_l(n)$ represents the l regular cubic partition pair. In this paper some infinite families of congruences and some Ramanujan-type congruence modulo 4 and 8 will be established for $B_2(n)$ and $B_3(n)$ such as

$$B_2\left(2.p^{2\alpha+1}.(pn+j)+\frac{3.p^{2\alpha+2}+1}{4}\right)\equiv 0\ (mod\ 8)$$

$$B_3\left(12.p^{2\alpha+1}.(pn+u)+\frac{3.p^{2\alpha+2}+1}{2}\right)\equiv 0\ (mod\ 4)$$

Keywords: Partition, cubic partition, congruences, dissection.

Introduction

A partition of a positive integer n is a non-increasing sequence of positive integer, known as parts, such that sum of the parts is n. It is denoted by $p(n)$ with $p(0) = 1$.
The generating function of $p(n)$ is given by

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty} = \frac{1}{f_1}$$

where for each complex number a, q with $|q| < 1$,

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1 - aq^n)$$

and $f_k = (q^k;q^k)_\infty$

Ramanujan [1, 2] investigated the arithmetic characteristics of $p(n)$. He found the three congruences for all $n \geq 0$

$$p(5n+4) \equiv 0(mod5)$$
$$p(7n+5) \equiv 0(mod7)$$
$$p(11n+6) \equiv 0(mod11)$$

Chan in several paper [3, 4, 5] initially examined the cubic partition of a positive integer n. It is denoted by $a(n)$ and the generating function for $a(n)$ is given by

$$\sum_{n=0}^{\infty} a(n)q^n = \frac{1}{f_1 f_2}$$

Motivated by Chan's work, Zhao and Zhong [6] explored cubic partition pairs, represented as $b(n)$, and the generating function is given by

$$\sum_{n=0}^{\infty} b(n)q^n = \frac{1}{f_1^2 f_2^2}$$

For each positive integer $l > 1$, a partition is said to be $l - regular$ if none of its part are divisible by l. It is denoted by $b_l(n)$ and the generating function for $b_l(n)$ is given by

$$\sum_{n=0}^{\infty} b_l(n)q^n = \frac{f_l}{f_1}$$

For example, $b_3(4) = 4$, given by $4,\ 2+2,\ 2+1+1,\ 1+1+1+1$
The $l - regular$ cubic partition pair is denoted by $B_l(n)$ and the generating function of $B_l(n)$ is defined by

$$\sum_{n=0}^{\infty} B_l(n)q^n = \frac{f_l^2 f_{2l}^2}{f_1^2 f_2^2} \qquad (1.1)$$

Naika & Nayaka [7], established several congruences for $B_l(n)$. Gireesh & Naika [8] also studied the arithmetic properties of $B_3(n)$ and $B_9(n)$ and proved several infinite families of congruences. Recently Wen [9] derived congruences modulo powers of 2 and 3 for $B_9(n)$.

Preliminaries.

In this section we will list some q $series$ identities and some 2-dissection and 3- dissection formulas which we will require to establish our result.
Ramanujan's general theta function [10, p.34, Equation 18.1] is defined by

$$f(a,b) = \sum_{n=0}^{\infty} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}}, \quad |ab| < 1$$

Using Jacobi's Triple product identity [10, p.35, Entry 19], $f(a,b)$ can be expressed as

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty$$

Three special cases of $f(a,b)$ are [10, p.36, Entry22]

$$\phi(q) = f(q,q) = \sum_{n=0}^{\infty} q^{k^2} = \frac{f_2^5}{f_1^2 f_4^2}$$

$$\psi(q) = f(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{k(k+1)}{2}} = \frac{f_2^2}{f_1^2}$$

$$f(-q) = f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^k q^{\frac{k(3k+1)}{2}} = f_1$$

Lemma 2.1 *The following 2-dissections hold*

$$\frac{1}{f_1^2}=\frac{f_8^5}{f_2^5 f_{16}^2}+2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8} \quad (2.1)$$

$$\frac{f_3^2}{f_1^2}=\frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}}+2q\frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}} \quad (2.2)$$

Proof. The proof of the identity (2.1) can be found in [11, Eq. (1.9.4)] and that of the identity (2.2) in [11, Eq. (30.10.4)]

Lemma 2.2 *We have, the following 3-dissection*

$$\frac{f_4}{f_1}=\frac{f_{12} f_{18}^4}{f_3^3 f_{36}^2}+q\frac{f_6^2 f_9^3 f_{36}}{f_3^4 f_{18}^2}+2q^2\frac{f_6 f_{18} f_{36}}{f_3^3} \quad (2.3)$$

Proof. The identity (2.3) is the Lemma 2.6 in [12].

Lemma 2.3 For *any odd prime p, we have*

$$\psi(q)=\sum_{n=0}^{\infty} q^{\frac{k^2+k}{2}} f(q^{\frac{p^2+(2k+1)p}{2}}, q^{\frac{p^2-(2k+1)p}{2}})+q^{\frac{p^2-1}{8}}\psi(q^{p^2})$$

Moreover, for, $0 \le k \le \frac{p-1}{2}$

$$\frac{k^2+k}{2} \neq \frac{p^2-1}{8} \pmod 8$$

Proof. Proof of the Lemma2.3 can be found in [13, Theorem 2.1]

Lemma 2.4 *For any prime* $p \ge 5$, *we have*

$$f_1=\sum_{\substack{k=-\frac{p-1}{2} \\ k\neq\pm\frac{p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right)+(-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f_{p^2}$$

$$\frac{\pm p-1}{6}=\begin{cases} \frac{p-1}{6} & p\equiv 1 \pmod 6 \\ \frac{-(p-1)}{6} & p\equiv -1 \pmod 6 \end{cases}$$

Where,

Furthermore, if $\frac{-(p-1)}{2} \le k \le \frac{p-1}{2}$ *and* $k \neq \frac{\pm p-1}{6}$,

Then $\frac{3k^2+k}{2} \not\equiv \frac{p^2-1}{24} \pmod p$.

Proof. Proof of the Lemma2.4 can be found in [13, Theorem 2.2]

Lemma 2.5. *For all primes p and all k, m,* $j \ge 1$, *we have*

$$f_{pm}^{p^{(k-1)j}} \equiv f_m^{p^k j} \pmod{p^k}$$

In particular the following congruences will be used frequently, so we may omit to refer this lemma in many occasions.

$$f_m^2 \equiv f_{2m} \pmod 2$$

$$f_m^4 \equiv f_{2m}^2 \pmod 4$$

$$f_m^8 \equiv f_{2m}^4 \pmod 8$$

$$f_m^{p^k} \equiv f_{pm}^{p^{k-1}} \pmod{p^k}$$

Main Results.

3.1. Congruences for $B_2(n)$

Theorem. 3.1.1. Let p be a prime such that $\left(\frac{-2}{p}\right)=-1$ and $1\le j\le p-1$. Then for all integers $n\ge 0$ and $\alpha\ge 0$, we have

$$\sum_{n=0}^{\infty} B_2\left(2.p^{2\alpha}.n+\frac{3.p^{2\alpha}+1}{4}\right)q^n \equiv 2\,\psi(q)\psi(q^4)\ (mod\ 8) \qquad (3.1)$$

$$B_2\left(2.p^{2\alpha+1}.(pn+j)+\frac{3.p^{2\alpha+2}+1}{4}\right)\equiv 0\ (mod\ 8) \qquad (3.2)$$

Proof. Setting $l=2$ in (1.1), we have

$$\sum_{n=0}^{\infty} B_2(n)q^n=\frac{f_2^2 f_4^2}{f_1^2 f_2^2}=\frac{f_4^2}{f_1^2} \qquad (3.3)$$

Employing (2.1) in (3.3), we obtain

$$\sum_{n=0}^{\infty} B_2(n)q^n=f_4^2\left(\frac{f_8^5}{f_2^5 f_{16}^2}+2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}\right) \qquad (3.4)$$

Extracting the terms that include q^{2n+1} from both sides (3.4), dividing both sides by q and then replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} B_2(2n+1)q^n=\frac{2f_2^4 f_8^2}{f_1^5 f_4}$$

using Lemma.2.5, we have

$$\sum_{n=0}^{\infty} B_2(2n+1)q^n=2\frac{f_2^4 f_8^2}{f_1^5 f_4}$$

$$\equiv 2f_1^3\ f_4^3\ (mod\ 8)$$

$$\equiv 2\ \psi(q)\psi(q^4)(\ mod\ 8) \qquad (3.5)$$

(3.5) is the $\alpha=0$ case of (3.1). Assume (3.1) is true some integer $\alpha\ge 0$. Employing Lemma.2.3 in (3.1), we obtain

$$\sum_{n=0}^{\infty} B_2\left(2.p^{2\alpha}.n+\frac{3.p^{2\alpha}+1}{4}\right)q^n \equiv\ 2\left[\sum_{x=0}^{\frac{p-3}{2}} q^{\frac{x^2+x}{2}} f(q^{\frac{p^2+(2x+1)p}{2}},q^{\frac{p^2-(2x+1)p}{2}})q^{\frac{p^2-1}{8}}\psi(q^{p^2})\right]$$

$$\left[\sum_{y=0}^{\frac{p-3}{2}} q^{2(y^2+y)} f\left(q^{2(p^2+(2y+1)p)},q^{2(p^2-(2y+1)p)}\right)+q^{\frac{p^2-1}{2}}\psi\left(q^{4p^2}\right)\right](mod\ 8) \qquad (3.6)$$

Consider the congruence

$$\frac{x^2+x}{2}+2(\ y^2+y)\ \equiv\frac{3(p^2-1)}{8}\ (mod\ p)$$

which is equivalent to

$$(2x+1)^2+2(2y+1)^2 \equiv 0 \pmod p$$

Since $\left(\frac{-2}{p}\right)=1$, the above congruence has only the solution $x=y=\frac{p-1}{2}$. Extracting the terms that include $q^{pn+\frac{3(p^2-1)}{8}}$ from both sides of (3.6), dividing both sides by $q^{\frac{3(p^2-1)}{8}}$ and then replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} B_2\left(2.p^{2\alpha}.\left(pn+\frac{3(p^2-1)}{8}\right)+\frac{3.p^{2\alpha}+1}{4}\right)q^n \equiv 2\,\psi(q^p)\psi(q^{4p}) \pmod 8$$

which yields

$$\sum_{n=0}^{\infty} B_2\left(2.p^{2\alpha+1}.n+\frac{3.p^{2\alpha+2}+1}{4}\right)q^n \equiv 2\,\psi(q^p)\psi(q^{4p}) \pmod 8 \qquad (3.7)$$

Similarly, extracting the terms that include q^{pn} from both sides (3.7) and replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} B_2\left(2.p^{2\alpha+2}.n+\frac{3.p^{2\alpha+2}+1}{4}\right)q^n \equiv 2\,\psi(q)\psi(q^4) \pmod 8$$

Above congruence is the $\alpha+1$ case of (3.1). Thus, by induction, (3.1) is true for all integer $\alpha \geq 0$. Finally, extracting the terms that include $q^{pn+j}, 1 \leq j \leq p-1$, from both sides of (3.7), we obtain (3.2).

Theorem. 3.1.2 *For all integer* $n \geq 0$ *and* $\alpha \geq 0$ *we have*

$$B_2(18n+8) \equiv 0 \pmod 4 \qquad (3.8)$$

$$B_2(18n+14) \equiv 0 \pmod 4 \qquad (3.9)$$

$$B_2\left(2.\ 3^{2\alpha+2}.\ n+\frac{3^{2\alpha+2}-1}{4}\right) \equiv B_2\,(2n) \pmod 4 \qquad (3.10)$$

Proof. Extracting the terms that include q^{2n} from (3.4) and thanks to Lemma 2.5, we have

$$\sum_{n=0}^{\infty} B_2(2n)q^n = \frac{f_2^2 f_4^5}{f_1^5 f_8^2} \equiv \frac{f_4}{f_1} \pmod 4 \qquad (3.11)$$

Utilizing (2.3), we have

$$\sum_{n=0}^{\infty} B_2(2n)q^n \equiv \left(\frac{f_{12}f_{18}^4}{f_3^3 f_{36}^2}+q\frac{f_6^2 f_9^3 f_{36}}{f_3^4 f_{18}^2}+2q^2\frac{f_6 f_{18} f_{36}}{f_3^3}\right) \pmod 4 \qquad (3.12)$$

Extracting the terms that include q^{3n+1} from (3.12), we have

$$\sum_{n=0}^{\infty} B_2(6n+2)q^n \equiv \frac{f_2^2 f_3^3 f_{12}}{f_1^4 f_6^2} \pmod 4 \equiv \frac{f_3^3 f_{12}}{f_6^2} \pmod 4 \qquad (3.13)$$

Extracting the terms that include q^{3n+1}, q^{3n+2} from (3.13), we obtain (3.8) and (3.9). Similarly, extracting the terms that include q^{3n} from (3.13), we obtain

$$\sum_{n=0}^{\infty} B_2(18n+2)q^n \equiv \frac{f_1^3 f_4}{f_2^2} \equiv \frac{f_4}{f_1} \pmod 4 \qquad (3.14)$$

From (3.11) and (3.14), we have (3.10) for $\alpha = 0$. Thus (3.10) holds for $\alpha = 0$. Applying induction on α, we obtain (3.10) for all integer $\alpha \geq 0$.

3.2 Congruence modulo 4 for $B_3(n)$

Theorem.3.2.1 *Suppose* p *is any prime with* $\left(\frac{-2}{p}\right) = -1$ *and* $1 \leq u \leq p-1$, *then for all integer* $n \geq 0$ *and* $\alpha \geq 0$, *we have*

$$B_3(4n+3) \equiv 0 \pmod 4 \qquad (3.15)$$

$$\sum_{n=0}^{\infty} B_3\left(12.p^{2\alpha}.n + \frac{3.\ p^{2\alpha}+1}{2}\right) q^n \equiv 2 f_1 f_2 \pmod 4 \qquad (3.16)$$

$$B_3\left(12.p^{2\alpha+1}.(pn+u) + \frac{3.p^{2\alpha+2}+1}{2}\right) \equiv 0 \pmod 4 \qquad (3.17)$$

Proof. Setting $l = 3$ in (1.1), we have

$$\sum_{n=0}^{\infty} B_3(n) q^n = \frac{f_3^2 f_6^2}{f_1^2 f_2^2} \qquad (3.18)$$

Using (2.2) in (3.18), we obtain

$$\sum_{n=0}^{\infty} B_3(n) q^n = \frac{f_6^2}{f_2^2}\left(\frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q\frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}}\right) \qquad (3.19)$$

Extracting the terms that include the odd power of q from (3.19), dividing by q and then replacing q^2 by q, we have

$$\sum_{n=0}^{\infty} B_3(2n+1) q^n = 2\frac{f_2 f_3^4 f_4 f_{12}}{f_1^6 f_6} = 2\frac{f_2 f_4 f_{12}}{f_6}\left(\frac{f_3^2}{f_1^2}\right)^2 \left(\frac{1}{f_1^2}\right)$$

Employing (2.1) and (2.2), we obtain

$$\sum_{n=0}^{\infty} B_3(2n+1) q^n = \frac{2 f_2 f_4 f_{12}}{f_6}\left(\frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q\frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}}\right)^2 \left(\frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}\right) \qquad (3.20)$$

From (3.20) we have

$$\sum_{n=0}^{\infty} B_3(2n+1) q^n = 2\frac{f_4^9 f_{12}^5 f_6 f_8^3}{f_2^{14} f_{24}^2 f_{16}^2} \pmod 4 \qquad (3.20)$$

Extracting those terms with odd power of q, we get (3.15).

Extracting those terms with even power of q, and then replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} B_3(4n+1) q^n = 2\frac{f_2^9 f_6^5 f_3 f_4^3}{f_1^{14} f_{12}^2 f_8^2} \equiv 2\, f_3 f_6 \pmod 4 \qquad (3.21)$$

Extracting those terms with power of q which are multiple of 3, we obtain

$$\sum_{n=0}^{\infty} B_3(12n+1) q^n \equiv 2\, f_1 f_2 \pmod 4 \qquad (3.22)$$

(3.22) is the $\alpha = 0$ case of (3.16). Assume (3.22) is true for some $\alpha \geq 0.$ Employing Lemma 2.4 in (3,22), we obtain

$$\sum_{n=0}^{\infty} B_3\left(12.p^{2\alpha}.n+\frac{3.\ p^{2\alpha}+1}{2}\right)q^n$$

$$\equiv 2\left[\sum_{\substack{k=-\frac{p-1}{2}\\ k\neq\pm\frac{p-1}{6}}}^{\frac{p-1}{2}}(-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}},-q^{\frac{3p^2-(6k+1)p}{2}}\right)\right.$$

$$\left.+(-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f_{p^2}\right] X \left[\sum_{\substack{m=-\frac{p-1}{2}\\ m\neq\pm\frac{p-1}{6}}}^{\frac{p-1}{2}}(-1)^m q^{3m^2+m} f\left(-q^{3p^2+(6m+1)p},-q^{3p^2-(6m+1)p}\right)\right.$$

$$\left.+(-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{12}} f_{2p^2}\right] (mod\ 4)$$

(3.23)
Consider the congruence

$$\frac{3k^2+k}{2}+(3m^2+m)\equiv\frac{p^2-1}{8}\ (mod\ p)$$

Which is equivalent to

$$(6k+1)^2+2\ (6m+1)^2\ \equiv 0\ (mod\ p)$$

Sine $\left(\frac{-2}{p}\right)=-1$, the last congruence has only the solution $k=m=\left(\frac{\pm p-1}{6}\right)$. Hence, extracting, the terms that include $q^{pn+\frac{p^2-1}{8}}$ from (3.23), dividing by $q^{\frac{p^2-1}{8}}$ and then replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} B_3\left(12.p^{2\alpha+1}.n+\frac{3.\ p^{2\alpha+2}+1}{2}\right)q^n \equiv 2\ f_p f_{2p}\ (mod\ 4) \qquad (3.24)$$

Extracting the terms that include q^{pn}, we obtain

$$\sum_{n=0}^{\infty} B_3\left(12.p^{2\alpha+2}.n+\frac{3.\ p^{2\alpha+2}+1}{2}\right)q^n \equiv 2\ f_1 f_2\ (mod\ 4)$$

Thus (3.16) is true for $\alpha+1$. Therefore, by induction that (3.16) always holds.
Finally extracting the terms that include q^{pn+u}, $1\leq u\leq p-1,$ from (3.24) we obtain (3.17).

Corollary. 3.2.2. For all integer $n\geq 0$, we have

$$B_3(12n+5)\ \equiv 0\ (mod\ 4) \qquad (3.25)$$

$$B_3(12n+9)\ \equiv 0\ (mod\ 4) \qquad (3.26)$$

Proof. Collecting the terms that include powers of q that are 1 modulo 3 and 2 modulo 3 from (3.21), we obtain (3.25) and (3.26).

Theorem 3.2.3 Suppose p is a prime with $\left(\frac{-3}{p}\right)=-1$ and w any integer with $1\leq w\leq p-1$, then for each integer $n\geq 0$ and $\gamma\geq 0$, we have

$$\sum_{n=0}^{\infty} B_3\left(4.p^{2\gamma}.n+\frac{7.\ p^{2\gamma}-1}{2}\right)q^n \equiv 4\,\psi(q^3)\psi(q^4)\ (mod\ 8) \qquad (3.27)$$

$$B_3\left(4.\ p^{2\gamma+1}.\ (pn+w)+\frac{7.p^{2\gamma+2}-1}{2}\right)\equiv 0\ (mod\ 8) \qquad (3.28)$$

Proof. From (3.20), extracting the terms that include q^{2n+1} and replacing q^2 by q, we have

$$\sum_{n=0}^{\infty} B_3(\,4n+3)q^n \equiv 4\frac{f_2^{11}f_3f_6^5f_8^2}{f_1^{14}f_4^3f_{12}^2}\ (mod\ 8) \qquad (3.29)$$

Thanks to Lemma 2.5, we have from (3.29)

$$\sum_{n=0}^{\infty} B_3(\,4n+3)q^n \equiv 4\ f_3^3f_4^3\ \equiv 4\ \ \psi(q^3)\psi(q^4)\ (mod\ 8) \qquad (3.30)$$

which is the $\gamma=0$ case of (3.27). Assume (3.27) holds for $\gamma\geq 0.$ Employing Lemma 2.3, we have

$$\sum_{n=0}^{\infty} B_3\left(4.p^{2\gamma}.n+\frac{7.\ p^{2\gamma}-1}{2}\right)q^n$$
$$\equiv 4\left[\sum_{k=0}^{\infty} q^{\frac{3(k^2+k)}{2}} f(q^{\frac{3(p^2+(2k+1)p)}{2}},q^{\frac{3(p^2-(2k+1)p)}{2}})q^{\frac{3(p^2-1)}{8}}\psi(q^{3p^2})\right] X$$

$$\left[\sum_{m=0}^{\infty} q^{2(m^2+m)} f(q^{2(p^2+(2m+1)p)},q^{2(p^2-(2m+1)p)})+q^{\frac{p^2-1}{2}}\psi(q^{4p^2})\right]$$

(3.31)
Consider the congruence

$$\frac{3(k^2+k)}{2}+2(m^2+m)\ \equiv \frac{7(p^2-1)}{8}\ (mod\ p)$$

which is equivalent to

$$(4k+2)^2+3(2m+1)^2\ \equiv 0\ (mod\ p)$$

For $\left(\frac{-3}{p}\right)=-1,$ the above congruence has only the solution $k=m=\frac{p-1}{2}$. So, extracting the terms that include $q^{pn+\frac{7(p^2-1)}{8}}$ from (3.31), dividing both sides by $q^{\frac{7(p^2-1)}{8}}$ and then replacing q^p by q, we have

$$\sum_{n=0}^{\infty} B_3\left(4.p^{2\gamma+1}.n+\frac{7.\ p^{2\gamma+2}-1}{2}\right)q^n \equiv 4\,\psi(q^{3p})\psi(q^{4p})\ (mod\ 8) \qquad (3.32)$$

Extracting the terms including q^{pn} from (3.32) and replacing q^p by q, we obtain

$$\sum_{n=0}^{\infty} B_3\left(4.p^{2\gamma+2}.n+\frac{7.\ p^{2\gamma+2}-1}{2}\right)q^n \equiv 4\,\psi(q^3)\psi(q^4)\ (mod\ 8)$$

which is the $\gamma+1$ case of (3.27). Hence, by induction on γ, (3.27) always holds.
Also, extracting the terms that include q^{pn+w}, $1\leq w\leq p-1$ from (3.32), we arrive at (3.28).

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