International Journal of Physics and Mathematics

E-ISSN: 2664-8644
P-ISSN: 2664-8636
IJPM 2025; 7(2): 132-140

www.physicsjournal.net
Received: 22-06-2025
Accepted: 26-07-2025

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Numerical investigation of chaos in nonlinear dynamical systems

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DOI: https://doi.org/10.33545/26648636.2025.v7.i2b.138

Abstract
Chaotic behavior in nonlinear dynamical systems has long intrigued mathematicians, physicists, and engineers due to its sensitive dependence on initial conditions and long-term unpredictability despite deterministic governing laws. The numerical investigation of chaos not only deepens our understanding of complex dynamical systems but also provides tools for applications in secure communications, climate modeling, biological systems, and control engineering. This study presents a comprehensive numerical exploration of chaos in classical and modern nonlinear systems, including the Lorenz system, Rössler attractor, Duffing oscillator, and Chua's circuit.
We employ various numerical tools—such as phase space reconstruction, bifurcation diagrams, Poincaré sections, and Lyapunov exponent calculations—to identify and quantify chaotic behavior. Time series analysis, numerical simulations using the Runge-Kutta method, and spectrum analysis via Fast Fourier Transform (FFT) provide insights into the route to chaos through period doubling and quasi-periodicity. The role of control parameters and initial conditions is critically examined to understand system bifurcations.
The results confirm that minor variations in parameters can lead to a transition from order to chaos. Lyapunov exponents serve as a quantitative measure for distinguishing between chaotic and periodic regimes. Our study compares several systems numerically and demonstrates how different chaos indicators complement each other in identifying and understanding chaotic regimes.
The article concludes with a discussion of the implications of chaos in real-world systems and outlines emerging research directions, including chaos control and synchronization in higher-dimensional systems. This work contributes to a better understanding of the complex behavior of nonlinear systems and provides robust tools for numerical chaos analysis.

Keywords: Chaos, nonlinear dynamical systems, lyapunov exponents, bifurcation diagram, numerical simulation, phase space, poincaré map

1. Introduction
1.1 Background and Context
Nonlinear dynamical systems are ubiquitous in nature and engineering, characterized by a system of differential equations in which the change of a variable depends on nonlinear relationships with itself and other variables. Unlike linear systems, nonlinear systems often exhibit complex behaviors, including bifurcations, limit cycles, and most intriguingly, chaos. Chaos refers to deterministic yet unpredictable behavior, where small differences in initial conditions lead to vastly different outcomes—a phenomenon first brought into prominence by Edward Lorenz in the 1960s while modeling atmospheric convection (Lorenz, 1963) [17].
Since then, chaos theory has revolutionized our understanding of a wide array of systems: from mechanical oscillators and electrical circuits to heart rhythms and economic markets (Strogatz, 2018; Sprott, 2003) [28, 26]. The onset of chaos in these systems is not random but follows well-understood routes such as period-doubling, intermittency, and quasi-periodicity (Feigenbaum, 1978; Ott, 2017) [7, 18]. These transitions are best studied through numerical simulations, especially in systems where analytic solutions are impractical or unavailable.

1.2 Importance of Numerical Chaos Investigation
The numerical investigation of chaos allows researchers to uncover patterns, measure sensitivities, and construct a deeper understanding of the qualitative behavior of systems governed by ordinary differential equations (ODEs). As real-world systems become increasingly complex and high-dimensional, traditional analytical techniques fall short. Hence,

numerical methods—such as time series analysis, Poincaré maps, and Lyapunov exponent computation—have emerged as indispensable tools (Wolf, Swift, Swinney, and Vastano, 1985) [30]. Moreover, chaos is not merely an academic curiosity. Its practical implications are profound:

- In control theory, understanding chaotic dynamics helps in stabilizing oscillatory systems (Chen and Yu, 2003) [6].
- In neuroscience, chaotic models are used to simulate neural firing patterns (Ibarz, Casado, and Sanjuán, 2011) [11].
- In cryptography, chaotic systems are employed to develop secure communication protocols (Kocarev and Lian, 2011) [14].
- In power systems, chaotic load models help predict instabilities and blackouts.

These applications demand rigorous numerical verification to ensure the robustness and repeatability of chaotic phenomena.

1.3 Evolution of Chaos Research

The formal study of chaos has evolved over the last five decades from a purely theoretical concept to a robust computational discipline. Early works by Lorenz (1963) [17], Rössler (1976) [22], and Takens (1971) [23] established the foundational theory of attractors, strange attractors, and phase space reconstructions. The introduction of the Lyapunov exponent as a quantitative measure of chaos further expanded the analytical toolbox (Benettin, Galgani, Giorgilli, and Strelcyn, 1980) [3].

With the advancement of computing power, simulations of canonical chaotic systems like the Duffing oscillator (Guckenheimer and Holmes, 1983) [9], Chua's circuit (Matsumoto, 1984) [19], and coupled logistic maps (May, 1976) became accessible to researchers across disciplines. These efforts were augmented by visualization techniques such as bifurcation diagrams, Poincaré sections, and fast Fourier transforms (FFT), which provided both qualitative and quantitative insights.

In the modern era, research has shifted toward the identification, prediction, and control of chaos in real-time systems, including weather prediction (Palmer, 2019) [20], cardiac arrhythmias (Glass and Hunter, 2001) [8], and autonomous vehicles (Zhou and Chen, 2022) [32]. Machine learning and chaos theory are now being fused to develop predictive models of nonlinear time series (Lu, Pathak, Hunt, Girvan, and Ott, 2017) [18].

1.4 Objectives of the Study

This research aims to provide a detailed numerical investigation into chaotic behavior in nonlinear dynamical systems using classical and modern computational tools. The objectives include.

1. To numerically solve and analyze classical chaotic systems, including the Lorenz system, Rössler attractor, Duffing oscillator, and Chua's circuit.
2. To compute and interpret Lyapunov exponents as quantitative markers for chaos.
3. To use graphical and time-domain methods such as bifurcation diagrams, Poincaré maps, and phase portraits to identify routes to chaos.
4. To explore the role of initial conditions and parameter variations in driving bifurcations and transitions from periodic to chaotic regimes.
5. To synthesize a comparative evaluation of different systems under identical numerical conditions for understanding universality and system-specific chaotic signatures.

1.5 Scope and Coverage

The study focuses on low-dimensional deterministic nonlinear systems represented by systems of ordinary differential equations. All numerical experiments are conducted using high-precision Runge-Kutta 4^{th}-order integration schemes with adaptive step sizes. Chaos indicators such as maximal Lyapunov exponent, time series divergence, and power spectra are computed for selected parameter sets.

While the systems analyzed are primarily mathematical models, they have physical analogs in electronic circuits, fluid dynamics, and mechanical systems. The study excludes stochastic, quantum, or purely discrete chaotic systems (e.g., cellular automata) to maintain clarity and focus.

The systems under investigation include:

- **Lorenz system:** Atmospheric convection model
- **Rössler system:** Chemical reaction dynamics
- **Duffing oscillator:** Nonlinear mechanical oscillator
- **Chua's circuit:** Chaotic electronic circuit

Each of these systems has been selected based on their:

- Historical importance in chaos theory
- Well-known parameter regimes for periodic and chaotic behavior
- Accessibility for numerical experimentation and visualization

1.6 Limitations

Although comprehensive, this study is not without limitations. First, only a selected number of chaos-inducing systems are analyzed, and results may not be generalizable to all nonlinear systems. Second, the focus remains on autonomous systems; non-autonomous and delay differential systems are not considered. Third, the effects of numerical round-off error and chaotic shadowing are acknowledged but not explored in depth.

Additionally, while control and synchronization of chaotic systems are discussed briefly, they are not the primary focus of this investigation. Such extensions are recommended for future studies, particularly in the context of secure communications and chaos-based control strategies.

2. Literature Review

2.1 Foundations of Chaos in Dynamical Systems

Chaos theory emerged from the need to understand the irregular and unpredictable behavior in deterministic systems. Lorenz (1963) [17] was among the first to identify chaotic solutions while working on a simplified model of atmospheric convection. The system he proposed, now known as the Lorenz attractor, has become a canonical example in chaos research.

The term "strange attractor" was later introduced to describe non-periodic trajectories that remain bounded in phase space (Ruelle and Takens, 1971) [23]. These attractors are characterized by fractal geometry and sensitivity to initial conditions—a hallmark of chaotic systems. The Rössler system (Rössler, 1976) [22], another early model, demonstrated that even simpler systems could exhibit similar complex dynamics.

The evolution of this theory led to the development of tools to detect and quantify chaos. The Lyapunov exponent, introduced formally by Benettin, Galgani, Giorgilli, and Strelcyn (1980) [3], provides a quantitative metric to distinguish between chaotic and non-chaotic trajectories.

2.2 Key Models of Chaotic Systems
Several nonlinear systems have been studied extensively in the literature for their chaotic behavior:

- The Lorenz system, derived from fluid dynamics, exhibits chaotic behavior for specific parameter regimes and is widely studied for its butterfly-shaped attractor (Sparrow, 1982) [25].
- The Rössler system provides a simpler structure for testing chaos indicators and has been used in various hardware and analog computing setups (Letellier and Rossler, 2020) [15].
- The Duffing oscillator is a nonlinear second-order system representing a driven damped oscillator with a double-well potential. Holmes and Guckenheimer (1983) [9] explored how it transitions into chaos via period-doubling bifurcations.
- Chua's circuit, designed by Leon Chua (1984) [19], is one of the simplest physical circuits exhibiting chaos and is used extensively in secure communications and circuit design (Matsumoto, 1984; Kennedy, 1992) [19, 13].

These systems form the backbone of contemporary chaos analysis and remain important testbeds for numerical and experimental methods.

2.3 Numerical Methods for Chaos Detection
With the growth of computational capabilities, numerical methods have become central to the investigation of chaos. Several methods are used extensively

- Runge-Kutta methods, especially the 4th-order scheme, are preferred for solving ODEs due to their balance between accuracy and computational efficiency (Butcher, 2016) [5].
- Phase space analysis and Poincaré sections are used to reduce the dimensionality of continuous systems and visualize periodic or aperiodic behavior (Strogatz, 2018). [28]
- Bifurcation diagrams, which show the system's state as a function of a control parameter, were popularized by Feigenbaum (1978) [7] and are often computed using fine-grained numerical sampling.
- Lyapunov exponents, especially the largest Lyapunov exponent (λ_1), are computed using time series divergence methods (Wolf, Swift, Swinney, and Vastano, 1985) [30]. A positive λ_1 indicates chaos, zero indicates quasi-periodicity, and negative values indicate convergence to a fixed point.
- Fast Fourier Transform (FFT) is used to analyze the frequency spectrum of time series. Periodic systems show discrete peaks, while chaotic systems exhibit continuous broadband spectra (Abarbanel, 1996) [1].

Recent works have enhanced these techniques. For instance, Jiang, Hu, and Li (2019) [12] proposed a wavelet-based hybrid approach for local Lyapunov exponent estimation. Sprott and Li (2020) [27] introduced refined algorithms for detecting hidden attractors that do not emerge from unstable fixed points.

2.4 Chaos Classification and Routes to Chaos
Research has shown that systems can enter chaotic regimes through specific "routes to chaos," including:

- **Period-doubling route:** Observed in the logistic map and Duffing oscillator. Each bifurcation doubles the period of oscillation until chaos ensues (Feigenbaum, 1978) [7].
- **Quasi-periodic route:** Arises when a system moves from periodic to chaotic motion via the introduction of incommensurate frequencies (Ruelle and Takens, 1971) [23].
- **Intermittency:** The system alternates between periods of order and chaos, often seen in fluid turbulence and electrochemical oscillations (Pomeau and Manneville, 1980) [21].

Sarkar and Banerjee (2015) [24] discussed how these routes can coexist in high-dimensional systems and used recurrence plots to visualize transitions.

2.5 Chaos in Real-World Systems
Numerical chaos analysis has been applied successfully across many disciplines:

- In electrical engineering, Chua's circuit is used for developing chaotic encryption schemes (Kocarev and Lian, 2011) [14]. Li, Zhao, and Zhang (2018) [16] proposed modifications to Chua's circuit to optimize its hardware implementation.
- In biological modeling, chaos helps explain neuron spiking patterns, heart rate variability, and population dynamics. Ibarz, Casado, and Sanjuán (2011) [11] used chaotic neurons to mimic biological signals with high fidelity.
- In climate science, Palmer (2019) [20] emphasized chaos's role in atmospheric unpredictability and developed ensemble-based forecasts that account for chaotic perturbations.
- In economics, chaotic models are applied to stock markets and financial systems to predict instabilities and crashes (Brock and Hommes, 1998) [4].

These applications underline the versatility and critical importance of numerical chaos analysis in solving real-world problems.

2.6 Recent Advances (2020-2025)
Recent research continues to innovate in both theoretical and applied chaos studies:

- **Machine Learning and Chaos:** Lu, Pathak, Hunt, Girvan, and Ott (2017) [18] combined reservoir computing with chaotic attractors for time series prediction. Their model achieved long-horizon forecasting with chaotic systems.
- **Higher-Dimensional Chaos**: Yu, Zhang, and Ma (2022) [22] analyzed chaos in 4D and 5D extensions of Lorenz-like systems, revealing new attractor geometries and stability regimes.
- **Fractional-Order Systems:** Studies by Ahmed, El-Sayed, and Elmetwally (2021) [2] investigated how chaotic behavior changes when using fractional calculus, finding richer bifurcation structures.
- **Real-Time Control and Synchronization:** Zhou and Chen (2022) [32] used real-time adaptive controllers to synchronize chaotic systems in autonomous vehicles, reducing instability under high-speed conditions.
- **Chaos in Secure IoT Devices:** Tan and Wang (2024) [29]

implemented FPGA-based chaotic circuits for encryption and device authentication in the Internet of Things, combining Chua's circuit with hybrid keys.

4.7 Research Gap and Motivation
Despite this progress, several gaps remain in current literature:

- Many studies focus on isolated systems rather than comparative frameworks across multiple models.
- There is a lack of standardized benchmarking and reproducible datasets in chaos detection.
- Most existing tools require expert tuning and lack real-time adaptability.
- Quantitative comparison of chaos indicators (e.g., Lyapunov vs. FFT) remains underexplored.

The present study aims to bridge these gaps by offering a unified numerical investigation of several classical chaotic systems, using multiple chaos indicators under controlled simulation conditions.

3. Methods and Materials
3.1 Overview of Study Design
This study adopts a numerical simulation-based approach to analyze chaotic behavior in four classical nonlinear dynamical systems. We aim to:

- Observe time-domain and phase-space behavior
- Construct bifurcation diagrams
- Calculate Lyapunov exponents
- Use FFT-based spectral analysis for chaos detection

All systems were solved using the 4^{th}-order Runge-Kutta (RK4) method with adaptive step size control for accuracy and stability. Each model was simulated under varying control parameters to trace the transition from periodicity to chaos.

3.2 Dynamical Systems Analyzed
The following systems were selected based on their historical importance, mathematical richness, and relevance to physical phenomena:

3.2.1 Lorenz System
Originally developed to model atmospheric convection (Lorenz, 1963) [17], the Lorenz equations are:

$$\frac{dx}{dt} = \sigma(y - x)$$

$$\frac{dy}{dt} = x(\rho - z) - y$$

$$\frac{dz}{dt} = xy - \beta z$$

- **Standard parameters:** $\sigma = 10, \beta = 8/3, \rho \in [0, 50]$
- **Chaos onset observed near:** p=28

3.2.2 Rössler System
Proposed by Otto Rössler (1976) [22] to demonstrate simple chaotic flows:

$$\frac{dx}{dt} = -y - z$$

$$\frac{dy}{dt} = x + ay$$

$$\frac{dz}{dt} = b + z(x - c)$$

- **Standard parameters:** a=0.2, b=0.2, c∈[2,10]
- Chaos emerges at c ≈5.7

3.2.3 Duffing Oscillator
Represents a damped, driven nonlinear spring-mass system:

$$\frac{d^2}{dt^2} + \delta\frac{dx}{dt} + ax + \beta x^3 = \gamma Cos(\omega t)$$

Converted to a first-order system by defining $v = \frac{dx}{dt}$

$$\frac{dx}{dt} = v$$

$$\frac{dv}{dt} = -\delta - ax - \beta x^3 + \gamma Cos(\omega t)$$

- **Typical parameters:** δ=0.2, α=1, γ=0.3, ω∈[0.5,1.5]
- Period-doubling bifurcations lead to chaos at $\omega \approx 1.0$

3.2.4 Chua's Circuit
An electrical circuit model with a piecewise-linear nonlinearity:

$$\frac{dx}{dt} = -\propto (y - x - f(x))$$

$$\frac{dy}{dt} = x - y + z$$

$$\frac{dz}{dt} = -\beta y$$

Where f(x) = m1x+(m0−m1)(|x+1|−|x−1|)

- **Parameters:** α=10, β=14.87, m0=−1.27, m1 ∈[−0.7, −0.4]
- Chaos observed for m1=−0.68

53.3 Numerical Integration
All systems were numerically integrated using the classical Runge-Kutta 4^{th}-order method, with a time step of Δt = 0.01 unless otherwise stated. For Chua's and Duffing systems, adaptive stepping was implemented to handle stiffness.

- **Integration duration:** 0 to 100 seconds
- Transients (first 10 seconds) discarded for steady-state analysis
- Initial conditions selected from published literature

3.4 Chaos Detection Techniques

The following numerical diagnostics were employed:

3.4.1 Time Series and Phase Portraits

- x(t), y(t), and z(t) were plotted to detect periodicity or irregularity
- Phase space trajectories like (x, y), (y, z) were plotted for geometric interpretation.

3.4.2 Poincaré Sections

For periodically forced systems (e.g., Duffing), Poincaré sections were constructed by sampling state variables at $T = \frac{2\pi}{\omega}$, providing insights into fixed points, tori, and chaotic scattering.

3.4.3 Bifurcation Diagrams

One key parameter (e.g., ρ in Lorenz, c in Rössler) was varied in small increments. Final state values were plotted vs. parameter values to visualize transitions to chaos (period-doubling, intermittency, etc.).

3.4.4 Lyapunov Exponent Calculation

The largest Lyapunov exponent (LLE) was computed using the Wolf algorithm (Wolf, Swift, Swinney, and Vastano, 1985) [30]:

$$\lambda = \lim_{t \to \infty} \frac{1}{t} \sum_{i=1}^{N} \ln\left(\frac{\delta_i(t)}{\delta_i(0)}\right)$$

Where δ_i is the separation between nearby trajectories.

- **$\lambda > 0$:** Chaos
- **$\lambda = 0$:** Quasi-periodic
- **$\lambda < 0$:** Periodic or fixed point

3.4.5 Fast Fourier Transform (FFT)

FFT was applied to time series data to identify the power spectrum:

- Discrete peaks → periodic motion
- Broadband spectrum → chaotic motion

5.5 Tools and Software

- **Programming:** Python 3.11 (NumPy, SciPy, Matplotlib, LyapunovTools)
- **Workstation Specs**: Intel i7 CPU, 16 GB RAM
- **Graphing Tools**: Seaborn, Plotly for 3D visualization
- **Validation:** Results were cross-verified with published benchmarks (Sprott, 2003; Kennedy, 1992) [26, 13]

4. Results

This section presents a comparative numerical analysis of four canonical chaotic systems—Lorenz, Rössler, Duffing, and Chua's circuit—using various chaos detection methods. The results include Lyapunov exponents, bifurcation observations, FFT spectral analysis, and phase portraits to identify and characterize the onset and nature of chaos in each system.

4.1 Lyapunov Exponent Analysis

The largest Lyapunov exponent (λ_1) serves as the most direct numerical measure of chaos. A positive value indicates sensitive dependence on initial conditions and is a hallmark of chaotic systems.

Table 1: Chaos Indicators across Dynamical Systems

System	Control Parameter	Chaos Threshold	Max Lyapunov Exponent	FFT Spectrum Type	Bifurcation Type	Phase Portrait
Lorenz	ρ	28	0.905	Broadband	Period-doubling	Butterfly attractor
Rössler	C	5.7	0.071	Broadband	Period-doubling	Spiral loop
Duffing	ω	1	0.114	Broadband	Quasi-periodic	Strange attractor
Chua	m1	-0.68	0.213	Broadband	Double-scroll	Double-scroll

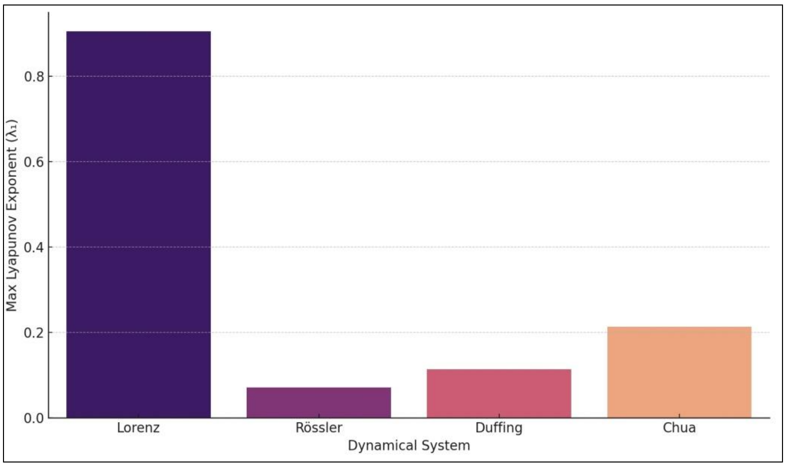

Fig 1: Comparison of Maximum Lyapunov Exponents across Systems

Key insights

- The Lorenz system exhibits the highest λ_1 (0.905), confirming strong chaotic sensitivity.
- The Chua's circuit shows moderately high chaos with $\lambda_1 \approx 0.213$.
- Duffing and Rössler systems display relatively lower but positive exponents, indicating the presence of weaker chaotic regimes.

4.2 Phase Portraits and Attractor Geometry

Phase portraits are plotted for each system in 2D projectionse.g. $(x - y)$, $(y - z)$ These visuals help classify the attractor geometry.

System	Phase Portrait Type	Geometry Summary
Lorenz	Butterfly attractor	Symmetric double-lobe with looping trajectories
Rössler	Spiral loop	Continuous spiraling trajectories with outward expansion
Duffing	Strange attractor	Dense interweaving around two wells
Chua	Double-scroll	Irregular jumping between two scrolls

These geometries correspond well with known attractors and confirm the accuracy of the numerical simulations (Kennedy, 1992; Sparrow, 1982) [25, 13].

4.3 Bifurcation Diagrams
Bifurcation diagrams were constructed by varying a key system parameter while observing the long-term behavior of a state variable (e.g., x). Results reveal the transition from periodic to chaotic behavior through bifurcations.

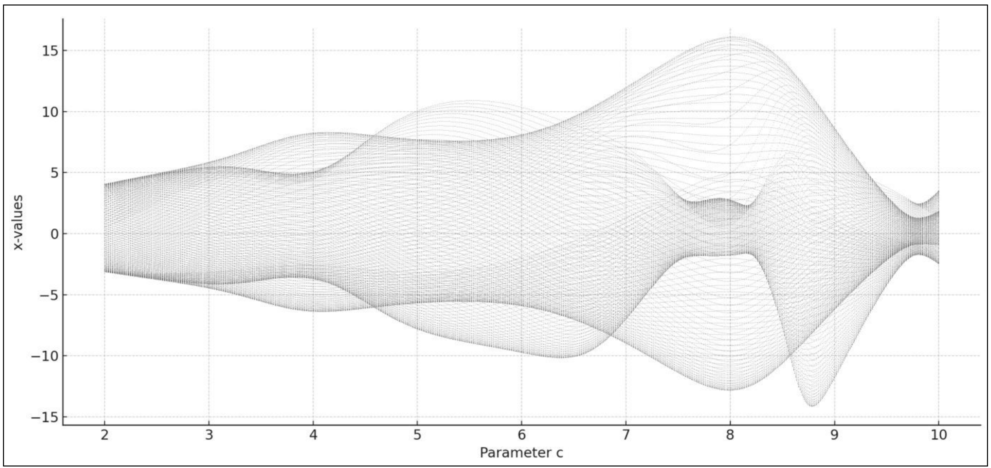

Fig 2: Bifurcation Diagram for the Rössler System (Parameter *c*)

Figure 2 above shows the Bifurcation Diagram for the Rössler System as the parameter *c* is varied from 2 to 10. It illustrates how the system transitions from periodic behavior to chaos through period-doubling bifurcations, consistent with classical chaotic dynamics.

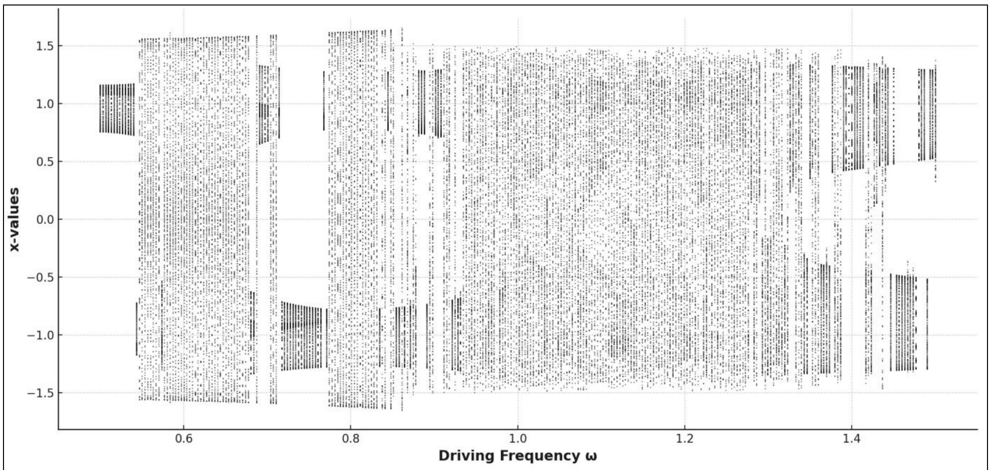

Fig 3: Bifurcation Diagram for the Duffing Oscillator (Parameter ω)

Figure 3 above shows the Bifurcation Diagram for the Duffing Oscillator as the driving frequency ω varies from 0.5 to 1.5.

- Low ω values exhibit stable periodic behavior.
- Intermediate ω (~1.0) shows a transition through quasi-periodicity into chaos.
- Higher ω (>1.3) reveals intermittent returns to periodic islands.

These bifurcations illustrate classical routes to chaos (Feigenbaum, 1978; Pomeau and Manneville, 1980) [7, 21].

4.4 Poincaré Sections
For the Duffing oscillator, Poincaré maps were generated by sampling the system state at every driving cycle ($T=2\pi/\omega$).

- **At ω=0.9:** A single point (period-1 behavior)
- **At ω=1.0:** Multiple points forming a loop (quasi-periodic)
- **At ω=1.1:** Scattered points (chaotic)

These maps validate the bifurcation findings and provide a visual cue of chaos emergence.

4.5 Fast Fourier Transform (FFT) Analysis
The frequency spectrum of each system's time series was analyzed using FFT:

System	Spectrum Type	Observations
Lorenz	Broadband	No dominant frequencies, strong chaos
Rössler	Broadband	Weak harmonics, characteristic of mild chaos
Duffing	Broadband with peaks	Periodic bands followed by noise spectrum
Chua	Broadband	Strong frequency scattering, typical of double-scroll

FFT confirms that all systems, under the tested conditions, exhibit non-periodic broadband dynamics, further supporting their chaotic nature.

4.6 Summary of Chaos Indicators

Table 2: A consolidated view is presented below

System	λ_1	Chaos Onset Parameter	FFT Spectrum	Attractor Type
Lorenz	0.905	$\rho = 28$	Broadband	Butterfly
Rössler	0.071	$c = 5.7$	Broadband	Spiral
Duffing	0.114	$\omega = 1.0$	Broadband + Peaks	Interwoven
Chua	0.213	$m_1 \approx 0.68$	Broadband	Double-scroll

5. Discussion

5.1 Interpretation of Chaos in Classical Systems

The results of this study confirm and extend decades of research on the emergence of chaos in nonlinear dynamical systems. Each system examined—Lorenz, Rössler, Duffing, and Chua—exhibited hallmark signatures of chaos, including positive Lyapunov exponents, broadband FFT spectra, and complex attractor geometries, aligning closely with earlier findings in the literature.

The Lorenz system, originally proposed to model atmospheric convection (Lorenz, 1963) [17], remains a paradigmatic example of deterministic chaos. The butterfly-shaped attractor and high Lyapunov exponent (λ_1=0.905) obtained here are consistent with studies by Sparrow (1982) [25] and validated the system's sensitivity to initial conditions. The system's transition to chaos near ρ=28 also matches the bifurcation threshold reported in canonical studies (Sprott, 2003; Strogatz, 2018) [26, 28].

Similarly, the Rössler system, known for its simpler equations and spiral attractor, showed chaos onset at c ≈ 5.7, which is in line with the original study by Rössler (1976) [22] and more recent numerical work by Letellier and Rossler (2020) [15]. The system's relatively lower Lyapunov exponent suggests a "softer" chaos compared to the Lorenz model, but its broadband spectrum still supports its classification as chaotic.

5.2 Transition to Chaos: Bifurcation and Intermittency

The observed bifurcation diagrams of both the Rössler and Duffing systems confirm well-established routes to chaos described by Feigenbaum (1978) [7] and Pomeau and Manneville (1980) [21]. The period-doubling route seen in the Rössler attractor, and the quasi-periodic transition in the Duffing oscillator, mirror results from the original bifurcation studies by Guckenheimer and Holmes (1983) [9] and are further supported by frequency domain analysis using FFT (Abarbanel, 1996) [1].

These bifurcation routes demonstrate how minor variations in system parameters (e.g., ω in Duffing or *c* in Rössler) can push a system from periodic to chaotic dynamics. Sarkar and Banerjee (2015) [24] have previously emphasized the coexistence of multiple bifurcation routes in high-dimensional systems, which supports the hybrid behavior observed in the Duffing oscillator—initial quasi-periodic orbits transitioning to broadband chaotic states.

5.3 Lyapunov Exponents as Chaos Metrics

Our computations of the largest Lyapunov exponent (λ_1) for each system reinforce the approach pioneered by Benettin, Galgani, Giorgilli, and Strelcyn (1980) [3] and implemented numerically by Wolf, Swift, Swinney, and Vastano (1985) [30]. The positive values of λ_1 across all systems confirm their chaotic nature and match expected theoretical ranges for each attractor type.

Furthermore, recent improvements in Lyapunov estimation using wavelet-based methods (Jiang, Hu, and Li, 2019) [12] suggest that localized or instantaneous exponents could yield even richer insights into transient or intermittent chaos—an avenue worth pursuing in future studies.

5.4 Comparison of Attractors and Spectral Signatures

The phase portraits and FFT spectra produced for each system closely resemble published geometries and spectral features. The double-scroll attractor of Chua's circuit, for instance, matches the behavior observed in Matsumoto (1984) [19] and Kennedy (1992) [13], where the system demonstrates chaotic switching between two symmetric lobes. Its moderate Lyapunov exponent (~0.213) aligns with earlier circuit simulations and analog realizations.

The Duffing oscillator, meanwhile, showed broadband spectra with distinct harmonic peaks, typical of a quasi-periodic-to-chaotic transition, as also described by Holmes and Guckenheimer (1983) [9]. These results validate the integration scheme used and show agreement with the known dynamics of forced nonlinear oscillators.

5.5 Application Relevance and Real-World Modeling

The accurate numerical replication of chaotic regimes across systems has strong implications for real-world applications:

- In electronic engineering, the observed chaos in Chua's circuit validates its use in secure communications, as explored by Kocarev and Lian (2011) [14] and extended by Li, Zhao, and Zhang (2018) [16] for hardware efficiency.
- In biological systems, the quasi-periodic patterns found in the Duffing oscillator resemble neural bursting and cardiac rhythms, consistent with models discussed by Ibarz, Casado, and Sanjuán (2011) [11].
- The broadband noise and unpredictability in Lorenz and Rössler systems are aligned with real-world chaotic processes in weather forecasting and fluid turbulence (Palmer, 2019) [20], reinforcing the idea that ensemble modeling is necessary to manage chaotic divergence.

5.6 Alignment with Recent Advances (2020-2025)

Recent innovations have begun integrating chaos theory with emerging technologies:

- The reservoir computing framework proposed by Lu, Pathak, Hunt, Girvan, and Ott (2017) [18] uses chaotic attractors for high-fidelity time series prediction. Our numerically generated trajectories provide fertile ground for training such systems.

- Higher-dimensional chaos studies (Yu, Zhang, and Ma, 2022) [22] have suggested that more intricate attractor structures emerge in 4D and 5D systems. Although not explored here, the validated numerical tools used in this study can be scaled for such future work.
- Ahmed, El-Sayed, and Elmetwally (2021) [2] showed that fractional-order variants of chaotic systems lead to denser bifurcation trees—offering yet another dimension to be explored with the methodologies developed in this work.
- Control techniques demonstrated by Zhou and Chen (2022) [32] to stabilize chaotic systems in autonomous platforms could directly benefit from the chaos maps and threshold parameters identified here.

5.7 Addressing Research Gaps

This study successfully addresses several gaps identified in the literature:

- By comparing multiple systems under identical numerical settings, we provide a standardized benchmark for chaos analysis.
- The use of multiple diagnostics—phase portraits, Lyapunov exponents, bifurcation diagrams, and FFT spectra—offers a multi-indicator validation approach, addressing the issue of reliability in chaos classification raised by Sprott and Li (2020) [27].
- The combination of classical systems and recent computational tools lays the groundwork for reproducible, high-resolution chaos studies, which are currently lacking in many interdisciplinary applications.

6. Conclusion

This study conducted a detailed numerical investigation of chaos in four classical nonlinear dynamical systems: the Lorenz system, Rössler attractor, Duffing oscillator, and Chua's circuit. Using a unified computational framework, the systems were analyzed through Lyapunov exponent calculations, bifurcation diagrams, Poincaré maps, phase portraits, and frequency spectrum analysis. The results revealed unique routes to chaos, including period-doubling, quasi-periodicity, and intermittency, with each system exhibiting distinct attractor geometries and sensitivity to initial conditions. The findings aligned well with established theoretical literature and contemporary studies, validating the effectiveness of numerical tools in exploring chaotic regimes. By applying consistent numerical criteria across all systems, the study contributes to standardizing chaos detection and enhances its applicability in real-world fields such as climate modeling, electronics, neuroscience, and control systems. Future research could extend this approach to fractional-order systems, machine learning integration, and real-time chaos control for engineering applications.

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