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On focal distances of ellipse revelation of hidden properties

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Abstract
Ellipse is one of the conic sections. It is an elongated circle. It is the locus of a point that moves in such a way that the ratio of its distance from a fixed point (called Focus) to its distance from a fixed line (called Directrix) equals to constant 'e' which is less than or equal to unity. According to the Keplar's law of Planetary Motion, the ellipse has very important role in geometry and the field of Astronomy, since in universe every planet is orbiting its star in an elliptical path and its star is as one of the foci. The objective of this research article is to establish few new theorems for mathematical properties for focal distances, focal chords. Totally 27 properties as theorems have been defined and proved with necessary derivations & drawings. The mathematical expressions of each theorem have also been given. These theorems will be very useful as reference to the research scholars to higher level research works in Geometry.

Keywords: Astronomy, planetary motion, ellipse, conic sections, focal distance, eccentricity, eccentric angle of ellipse, conjugate axis, transverse axis, auxiliary circle, tangent, normal, focus, vertex, conjugate diameter

Introduction

An *ellipse* is the set of all points in a plane such that the sum of the distances from two fixed points called *foci* is a given constant. Things that are in the shape of an ellipse are said to be elliptical. In the 17th century, a mathematician Mr. Johannes Kepler discovered that the orbits along which the planets travel around the Sun are ellipses with the Sun at one of the foci, in his First law of planetary motion. Later, Isaac Newton explained that this as a corollary of his law of universal gravitation. One of the physical properties of ellipse is that sound or light rays emanating from one focus will reflect back to the other focus. The longest and shortest diameters of an ellipse is called *Major axis* and *Minor axis* respectively. The *eccentricity* of an ellipse, usually denoted by ε or e, is the ratio of the distance between the two foci to the length of the major axis. A straight line passing an ellipse and touching it at just one point is called *tangent*. A straight line which passing through the centre of ellipse is called *diameter*. A straight line that passing through the centre of the parallel lines to the diameter ellipse is called *conjugate diameter*. The distances between foci and a point of contact by tangent with ellipse is called *focal distances* at that point. There are some existing properties of ellipse such as "sum of focal distances is a constant" (called focal constant), reflection property of focus of the ellipse, etc.

Theorem- 1

If any two chords PQ & RS which are perpendicular to each other passing through one of its foci F_1 of an ellipse, then the sum of reciprocals of product of PF_1 & QF_1 and RF_1 & SF_1 is equal to $(a^2 + b^2)/b^4$ which is an arbitrary constant. where, a & b are semi major axis and semi-minor axis respectively. This can be written mathematically as

$$\left(\frac{1}{PF_1 \times QF_1}\right) + \left(\frac{1}{RF_1 \times SF_1}\right) = \left(\frac{a^2 + b^2}{b^4}\right)$$

Now an attempt has been made by the Author to develop mathematical properties as theorems for focal distances using parametric equation of ellipse. The new properties have been derived mathematically and proven. The geometrical properties, which have been defined in this research article is very useful for those doing research works or further study in the field of Astronomy, Conics and Euclidean geometry, since this is also one of the important properties of an ellipse. This may also be very important to scientists who work in the field of Optics.

Proof of the theorem- 1

A tangent is drawn at point P of an ellipse. F_1 & F_2 are the foci of the ellipse. a & b are the semi-major axis & semi-minor axis respectively; O is centre of the ellipse. points Q & R are the projection of point P on transverse axis (Christopher Clapham & James Nicolson, 2009) [2] & conjugate axis (Borowski E.J & Borwein J.M, 1991) [1]. Let, $\angle QF_1P = \beta°$, PF_1 & PF_2 are pair focal distances (Borowski E.J & Borwein J.M, 1991) [1] at point P. Points T & S are intersection of tangent with transverse axis & conjugate axis respectively.

Referring fig.1, we know already that $OF_1 = \sqrt{a^2 - b^2}$ which is called as 'Numerical eccentricity' of ellipse (Borowski E.J & Borwein J.M, 1991) [1].

Let, PF_1 is the focal distance and $\angle AF_1P = \beta°$.

$$PF_1 = \frac{b^2}{a - \left[\sqrt{a^2 - b^2}\cos(\beta°)\right]} \quad (Ara.Kalaimaran, May, 2025) \qquad ----------------(1.1)$$

If $\angle AF_1P = \beta°$, QF_1 is perpendicular to PF_1. $\therefore \angle AF_1R = 90° + \beta°$

$$\therefore RF_1 = \frac{b^2}{a - \left[\sqrt{a^2 - b^2}\cos(90° + \beta°)\right]}$$

$$\therefore RF_1 = \frac{b^2}{a + \left[\sqrt{a^2 - b^2}\sin(\beta°)\right]} \qquad ----------------------------(1.2)$$

If $\angle AF_1P = \beta°$, QF_1 & PF_1 are collinear. $\therefore \angle AF_1Q = (180° + \beta°)$

$$\therefore QF_1 = \frac{b^2}{a - \left[\sqrt{a^2 - b^2}\cos(180° + \beta°)\right]}$$

$$\therefore QF_1 = \frac{b^2}{a + \left[\sqrt{a^2 - b^2}\cos(\beta°)\right]} \qquad ------------------------------(1.3)$$

$\angle AF_1P = \beta°$, QF_1 is perpendicular to PF_1. $\therefore \angle AF_1S = 270° + \beta°$

$$SF_1 = \frac{b^2}{a - \left[\sqrt{a^2 - b^2}\cos(270 + \beta°)\right]}$$

$$\therefore SF_1 = \frac{b^2}{a - \left[\sqrt{a^2 - b^2}\sin(\beta°)\right]} \qquad ------------------------------(1.4)$$

Multiplying eqns. (1.1) and (1.3),

$$PF_1 \times QF_1 = \left(\frac{b^2}{a - \left[\sqrt{a^2 - b^2}\cos(\beta°)\right]}\right) \times \left(\frac{b^2}{a + \left[\sqrt{a^2 - b^2}\cos(\beta°)\right]}\right)$$

Simplifying the above eqn,

$$\therefore PF_1 \times QF_1 = \frac{b^4}{a^2 \sin^2 \beta° + b^2 \cos^2 \beta°}$$

$$\therefore \frac{1}{PF_1 \times QF_1} = \frac{a^2 \sin^2 \beta° + b^2 \cos^2 \beta°}{b^4} \quad ----------------------- (1.5)$$

Multiplying eqns. (1.2) and (1.4),

$$RF_1 \times SF_1 = \left(\frac{b^2}{a + \left[\sqrt{a^2 - b^2}\,\sin(\beta°)\right]}\right) \times \left(\frac{b^2}{a - \left[\sqrt{a^2 - b^2}\,\sin(\beta°)\right]}\right)$$

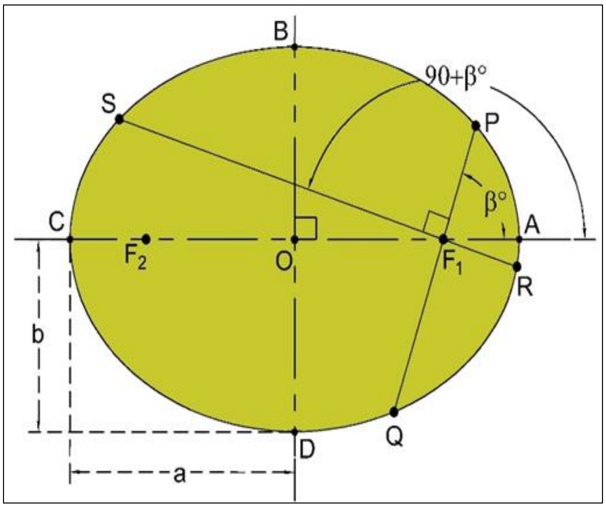

Fig 1

$$\therefore \frac{1}{PF_1 \times QF_1} + \frac{1}{RF_1 \times SF_1} = \left(\frac{a^2(\sin^2 \beta° + \cos^2 \beta°) + b^2(\cos^2 \beta° + \sin^2 \beta°)}{b^4}\right)$$

$$\therefore \left(\frac{1}{PF_1 \times QF_1}\right) + \left(\frac{1}{RF_1 \times SF_1}\right) = \frac{a^2 + b^2}{b^4} \qquad 1.7$$

Where, $\frac{a^2 + b^2}{b^4}$ is an arbitrary constant ($Borowski\ E.J$ & $Borwein\ J.M$ 1991).

Eqns. (1.7) is mathematical expressions of the theorem.

Theorem- 2

If F_1 & F_2 are foci of an ellipse respectively, OP_1 & OP_2 are semi-diameter & semi-conjugate diameter respectively, OQ_1 & OR_1 are parametric co-ordinates of point P_1 similarly OQ_2 & OR_2 are parametric co-ordinates of point P_2, then

$$\overline{P_2F_1} \times \overline{P_2F_2} = \overline{OP_1}^2$$

Proof of the theorem- 2

Referring fig.2, OP_1 & OP_2 are pair semi-conjugate diameters each other. If parametric equation of $P_1 = a\cos(\theta°), b\sin(\theta°)$ then parametric equation of P_2 will be $a\cos(90-\theta°), b\sin(90-\theta°)$. $\overline{P_1F_1}$ and $\overline{P_1F_2}$ are focal distances of P_1 from F_1 & F_2 respectively, θ° is the eccentric angle *(Henry Burchard Fine & Henry Dallas Thompson, 1909)*

$$\text{Eqn.}\ (1.1) \Rightarrow \overline{P_1F_1} = a - \left(\sqrt{a^2 - b^2} \times \cos\theta°\right) \qquad 2.1$$

$$\overline{P_1F_2} = a + \left(\sqrt{a^2 - b^2} \times \cos\theta°\right) \quad \text{(Ara. Kalaimaran, May-2025)} \qquad 2.2$$

In right-triangle $F_1Q_2P_2$, $\angle F_1Q_2P_2 = 90°$

$\therefore$ parametric equation of $P_2 = a\sin(\theta°), b\cos(\theta°)$

$$\overline{P_2F_1} = a - \left(\sqrt{a^2 - b^2} \times \sin\theta°\right) \quad 2.3$$

$$\overline{P_2F_2} = a + \left(\sqrt{a^2 - b^2} \times \sin\theta°\right) \quad 2.4$$

Multiplying eqns. (2.3) and (2.4),

$$\overline{P_2F_1} \times \overline{P_2F_2} = \left[a - \left(\sqrt{a^2 - b^2} \times \sin\theta°\right)\right] \times \left[a + \left(\sqrt{a^2 - b^2} \times \sin\theta°\right)\right]$$

$$\therefore \overline{P_2F_1} \times \overline{P_2F_2} = a^2 - (a^2 - b^2) \times \sin^2\theta°$$

Simplifying the above eqn.,

$$\therefore \overline{P_2F_1} \times \overline{P_2F_2} = a^2\cos^2(\theta°) + b^2\sin^2(\theta°) \quad 2.5$$

$$\text{Eqn. } (11.3) \Rightarrow \overline{OP} = \sqrt{a^2\cos^2(\theta°) + b^2\sin^2(\theta°)}$$

(Ara. Kalaimaran, May-2025)

$$\therefore \overline{OP}^2 = a^2\cos^2(\theta°) + b^2\sin^2(\theta°) \quad 2.6$$

Comparing eqns. (2.5) and (2.6),

$$\overline{P_2F_1} \times \overline{P_2F_2} = \overline{OP}^2 \quad 2.7$$

Eqn. (2.7) is mathematical expression of the theorem.

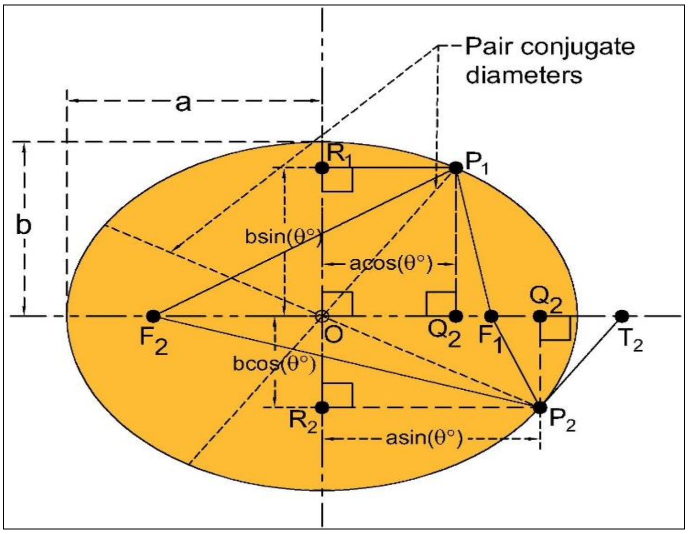

Fig 2

Theorem- 3

If a tangent is drawn to ellipse at point P, the tangent intersects the transverse axis at point T and intersect the conjugate axis at S, similarly a normal is drawn to the ellipse through the point P, the normal intersect the transverse axis at point N and intersect the conjugate axis at M, then the product of PT, PS, PM, PN is equal to unity. This can be written in mathematical form as

$$PT \times PS \times PM \times PN = (PF_1 \times PF_2)^2$$

Proof of the theorem- 3

Referring fig.3, a tangent is drawn at point P of an ellipse. F_1 & F_2 are the foci of the ellipse. a & b are the semi-major axis & semi-minor axis respectively, point O is centre of the ellipse. Points T & S are the point of intersection of tangent with transverse axis & conjugate axis respectively, points Q & R are the projection of point P on transverse axis & conjugate axis. QT & QN are sub-tangent (Borowski E.J & Borwein J.M, 1991) [1] & sub-normal (Mark D Licker, 2003) [1] with respect to transverse axis, RS & RM are sub-tangent & sub-normal with respect to conjugate axis, OQ & OR are the abscissa (Borowski E.J & Borwein J.M, 1991) [1] & ordinate (Borowski E.J & Borwein J.M, 1991) [1] of point tangent point P on the ellipse respectively, OT & OS are x-intercept and y-intercept of a tangent drawn at point P for an ellipse respectively.

1. Let, $\angle UOQ = \theta°$, hence $\angle QTU = (90° - \theta°), \angle SOV = (90° - \theta°), \quad \angle OSV = \theta°$

$OU = a, OV = b$, parametric equation of ellipse $P\ (x, y) = a\cos(\theta°), b\sin(\theta°)$

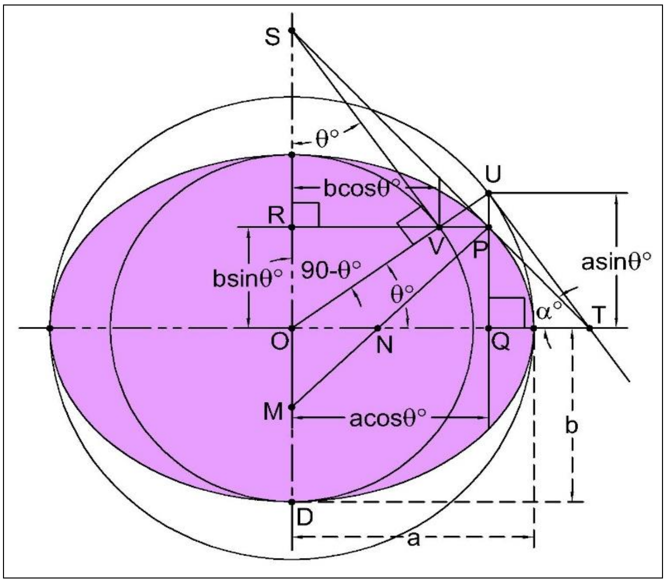

Fig 3

$\therefore OQ = a\cos(\theta°)$ 3.1

$\therefore OR = PQ = b\sin(\theta°)$ 3.2

$\therefore UQ = a\sin(\theta°)$ 3.3

In right Δ OQU, $\angle QTU = (90° - \theta°)$

$$\sin(\theta°) = \frac{UQ}{OU}$$

$\therefore UQ = OU \times \sin(\theta°)$

$\therefore UQ = a \times \sin(\theta°)$ 3.4

In right Δ QTU, $\angle QTU = (90° - \theta°)$

$$\tan(90° - \theta°) = \frac{UQ}{QT}$$

Simplifying the above eqn.,

$\therefore QT = UQ \times \tan(\theta°)$

Substituting eqn. (3.3) in above,

$$\therefore QT = a\sin(\theta^\circ) \times \frac{\sin(\theta^\circ)}{\cos(\theta^\circ)}$$

$$\therefore QT = \frac{a\sin^2(\theta^\circ)}{\cos(\theta^\circ)} \qquad 3.5$$

In fig. 3, $OT = OQ + QT$

Substituting eqn. (3.1) & (3.5) in above,

$$\therefore OT = a\cos(\theta^\circ) + \left(\frac{a\sin^2(\theta^\circ)}{\cos(\theta^\circ)}\right)$$

Simplifying the above eqn.,

$$\therefore OT = \frac{a}{\cos(\theta^\circ)} \qquad 3.6$$

In right Δ ORV, $\angle ROV = (90^\circ - \theta^\circ)$

$$\tan(90^\circ - \theta^\circ) = \frac{VR}{OR}$$

$$\therefore VR = \frac{OR}{\tan(\theta^\circ)}$$

Substituting eqn. (3.2) in above,

$$\therefore VR = \frac{b\sin(\theta^\circ)}{\tan(\theta^\circ)}$$

$$\therefore VR = b\cos(\theta^\circ) \qquad 3.7$$

In right Δ SRV, $\angle RSV = (\theta^\circ)$

$$\tan(\theta^\circ) = \frac{VR}{RS}$$

$$\therefore RS = \frac{VR}{\tan(\theta^\circ)}$$

Substituting eqn. (3.7) in above,

$$\therefore RS = \frac{b\cos(\theta^\circ)\cos(\theta^\circ)}{\sin(\theta^\circ)}$$

$$\therefore RS = \frac{b\cos^2(\theta^\circ)}{\sin(\theta^\circ)} \qquad 3.8$$

In fig. 3, $OS = OR + RS$

Substituting eqns. (3.2) and (3.8) in above,

$$OS = b\sin(\theta^\circ) + \left(\frac{b\cos^2(\theta^\circ)}{\sin(\theta^\circ)}\right)$$

Simplifying the above eqn.,

$$\therefore OS = \frac{b}{\sin(\theta°)} \qquad 3.9$$

Multiplying eqns. (3.1) and (3.6),

$$OQ \times OT = a\cos(\theta°) \times \left(\frac{a}{\cos(\theta°)}\right)$$

$$\therefore OQ \times OT = a^2 \qquad 3.10$$

Multiplying eqns. (3.2) and (3.9),

$$OR \times OS = b\cancel{\sin(\theta°)} \times \frac{b}{\cancel{\sin(\theta°)}}$$

$$\therefore OR \times OS = b^2 \qquad 3.11$$

According to Pythagoras theorem on right Δ TQP,

$$PT^2 = QT^2 + PQ^2$$

Substituting eqns. (3.5) and (3.2) in above,

$$PT^2 = \left(\frac{a\sin^2(\theta°)}{\cos(\theta°)}\right)^2 + [b\sin(\theta°)]^2$$

Simplifying the above eqn.,

$$\therefore PT^2 = \tan^2(\theta°)[a^2\sin^2(\theta°) + b^2\cos^2(\theta°)]$$

$$\therefore PT = \tan(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2\cos^2(\theta°)} \qquad 3.12$$

In right Δ TQP, $\tan(\alpha°) = \frac{PQ}{QT}$

Substituting eqn. (3.2) & (3.5) in above

$$\tan(\alpha°) = b\sin(\theta°) \div \left(\frac{a\sin^2(\theta°)}{\cos(\theta°)}\right)$$

Simplifying the above eqn.,

$$\tan(\alpha°) = \frac{b}{a\tan(\theta°)} \qquad 3.13$$

Referring fig. 3, in right triangle PQN, $\tan(\alpha°) = \frac{NQ}{PQ}$

$$\therefore NQ = PQ \times \tan(\alpha°)$$

$$\therefore NQ = b\sin(\theta°) \times \tan(\alpha°),$$

Substituting eqn. (3.13) in above,

$$\therefore NQ = b\sin(\theta°) \times \left(\frac{b}{a\tan(\theta°)}\right)$$

$$\therefore \mathrm{NQ} = \frac{b^2\cos(\theta°)}{a} \quad 3.14$$

According to Pythagoras theorem on right Δ NQP,

$$\mathrm{PN}^2 = \mathrm{NQ}^2 + \mathrm{PQ}^2$$

Substituting eqn. (3.14) and (3.2) in above eqn.,

$$\mathrm{PN}^2 = \left(\frac{b^2\cos(\theta°)}{a}\right)^2 + \left(b\sin(\theta°)\right)^2$$

Simplifying the above eqn.,

$$\mathrm{PN}^2 = \frac{b^2}{a^2}\left(a^2\sin^2(\theta°) + b^2\cos^2(\theta°)\right)$$

$$\therefore \mathrm{PN} = \frac{b\sqrt{a^2\sin^2(\theta°) + b^2\cos^2(\theta°)}}{a} \quad 3.15$$

Multiplying (12.1) & (12.2),

$$\mathrm{PN} \times \mathrm{PT} = \frac{[a^2\sin^2(\theta°) + b^2\cos^2(\theta°)] \times b\sin(\theta°)}{a\cos(\theta°)} \quad 3.16$$

$$\text{Eqn.}\,(1.1) \Longrightarrow \mathrm{PF}_1 = a - \left(\sqrt{a^2 - b^2} \times \cos\theta°\right)$$

$$\text{Eqn.}\,(2.1) \Longrightarrow \mathrm{PF}_2 = a + \left(\sqrt{a^2 - b^2} \times \cos\theta°\right)$$

Multiplying these two,

$$\mathrm{PF}_1 \times \mathrm{PF}_2 = \left[a - \left(\sqrt{a^2 - b^2} \times \cos\theta°\right)\right] \times a + \left(\sqrt{a^2 - b^2} \times \cos\theta°\right)$$

Simplifying the above eqn.,

$$\mathrm{PF}_1 \times \mathrm{PF}_2 = a^2\sin^2(\theta°) + b^2\cos^2(\theta°) \quad 3.17$$

Substituting (3.17) in eqn. (3.16),

$$\therefore \mathrm{PN} \times \mathrm{PT} = \frac{[\mathrm{PF}_1 \times \mathrm{PF}_2] \times b\sin(\theta°)}{a\cos(\theta°)} \quad 3.18$$

$$\mathrm{ON} = \mathrm{OQ} - \mathrm{NQ}$$

Substituting eqns. (3.1) & (3.14) in above eqn.,

$$\mathrm{ON} = a\cos(\theta°) - \left(\frac{b^2\cos(\theta°)}{a}\right)$$

Simplifying the above eqn.,

$$\therefore \mathrm{ON} = \frac{(a^2 - b^2)\cos(\theta°)}{a} \quad 3.19$$

Referring fig.3,

In right triangle NOM, $\angle \mathrm{NOM} = \alpha°$. In this right triangle, $\tan(\alpha°) = \frac{\mathrm{ON}}{\mathrm{OM}}$

$$\therefore \mathrm{OM} = \frac{\mathrm{ON}}{\tan(\alpha°)}$$

Substituting eqn. (3.18) & (3.13) in above eqn.,

$$\therefore \mathrm{OM} = \left(\frac{(a^2 - b^2)\cos(\theta°)}{a}\right) \times \frac{a\tan(\theta°)}{b}$$

Simplifying the above eqn.,

$$\therefore \mathrm{OM} = \frac{(a^2 - b^2)\sin(\theta°)}{b} \qquad 3.20$$

In right-triangle NOM, $\mathrm{MN}^2 = \mathrm{OM}^2 + \mathrm{ON}^2$

Substituting eqn. (3.18) and eqn. (3.20) in above, we get

$$\mathrm{MN}^2 = \left(\frac{(a^2 - b^2)\sin(\theta°)}{b}\right)^2 + \left(\frac{(a^2 - b^2)\cos(\theta°)}{a}\right)^2$$

Simplifying the above eqn.,

$$\therefore \mathrm{MN}^2 = (a^2 - b^2)^2\left(\frac{a^2\sin^2(\theta°) + b^2\cos^2(\theta°)}{a^2b^2}\right)$$

$$\therefore \mathrm{MN} = \frac{(a^2 - b^2)\sqrt{a^2\sin^2(\theta°) + b^2\cos^2(\theta°)}}{ab} \qquad 3.20$$

Referring fig.3, $\mathrm{PM} = \mathrm{PN} + \mathrm{MN}$
Adding eqns. (3.15) and (3.20)

$$\therefore \mathrm{PM} = \mathrm{PN} + \mathrm{MN} = \left(\frac{(a^2 - b^2)\sqrt{a^2\sin^2(\theta°) + b^2\cos^2(\theta°)}}{ab}\right) + \left(\frac{b\sqrt{a^2\sin^2(\theta°) + b^2\cos^2(\theta°)}}{a}\right)$$

Simplifying the above eqn.,

$$\therefore \mathrm{PM} = \frac{a\sqrt{a^2\sin^2(\theta°) + b^2\cos^2(\theta°)}}{b} \qquad 3.21$$

According to Pythagoras theorem on right Δ SRP,

$$\therefore \mathrm{PS}^2 = \mathrm{RS}^2 + \mathrm{OQ}^2$$

Substituting eqns. (3.8) & (3.1) in above,

$$\therefore \mathrm{PS}^2 = \left(\frac{b\cos^2(\theta°)}{\sin(\theta°)}\right)^2 + [a\cos(\theta°)]^2$$

Simplifying the above eqn.,

$$\therefore \mathrm{PS}^2 = \cot^2(\theta°)\,[a^2\sin^2(\theta°) + b^2\cos^2(\theta°)]$$

$$\therefore PS = \frac{\sqrt{a^2\sin^2(\theta°) + b^2\cos^2(\theta°)}}{\tan(\theta°)} \tag{3.22}$$

$$PM \times PS = \frac{[a^2\sin^2(\theta°) + b^2\cos^2(\theta°)] \times a\cos(\theta°)}{b\sin(\theta°)}$$

Substituting eqn. (3.17) in above eqn.,

$$\therefore PM \times PS = \frac{[PF_1 \times PF_2] \times a\cos(\theta°)}{b\sin(\theta°)} \tag{3.23}$$

Multiplying (3.17) and (3.24)

$$PN \times PT \times PM \times PS = \left(\frac{[PF_1 \times PF_2] \times \cancel{b\sin(\theta°)}}{\cancel{a\cos(\theta°)}}\right) \times \left(\frac{[PF_1 \times PF_2] \times \cancel{a\cos(\theta°)}}{\cancel{b\sin(\theta°)}}\right)$$

$$\therefore PT \times PS \times PM \times PN = (PF_1 \times PF_2)^2 \tag{3.24}$$

Eqn. (3.24) is mathematical expression of the theorem.

Theorem- 4

If points F_1 & F_2 are foci of an ellipse and PF_1 & PF_2 are pair focal distances, QT is sub-tangent and QN is sub-normal on transverse axis through tangent point P, then the product of PF_1 & PF_2 is equal to product of abscissa of point P and the sum of sub tangent and sub-normal. In mathematical form

Proof of the theorem- 4

Referring the fig. 13, a tangent is drawn at point P of an ellipse. F_1 & F_2 are the foci of the ellipse. a & b are the semi-major axis & semi-minor axis respectively, point O is centre of the ellipse. points T & S are the point of intersection of tangent with transverse axis & conjugate axis respectively, points Q & R are the projection of point P on transverse axis & conjugate axis. QT & QN are sub-tangent & sub-normal with respect to transverse axis, RS & RM are sub-tangent & sub-normal with respect to conjugate axis, OQ & OR are the abscissa & ordinate of point tangent point P on the ellipse respectively. OT & OS are x-intercept and y-intercept of a tangent drawn at point P for an ellipse respectively, ON & OM are x-intercept and y-intercept of a normal drawn at point P respectively.

$$\text{Eqn.}(3.6) \Longrightarrow OT = \frac{a}{\cos(\theta°)} \tag{4.1}$$

$$\text{Eqn.}(3.18) \Longrightarrow ON = \frac{(a^2 - b^2)\cos(\theta°)}{a} \tag{4.2}$$

Referring the fig.3, $NT = OT - ON$

Substituting the eqn. (4.1) and (4.2) in above

$$NT = \left(\frac{a}{\cos(\theta°)}\right) - \left(\frac{(a^2 - b^2)\cos(\theta°)}{a}\right)$$

Simplifying the above eqn.,

$$\therefore NT = \frac{a^2\sin^2(\theta°) + b^2\cos^2(\theta°)}{a\cos(\theta°)} \tag{4.3}$$

Substituting, $a\cos(\theta°) = OQ$ & substituting eqn. (3.17) in above eqn.,

$$NT = \frac{PF_1 \times PF_2}{OQ}$$

$$\therefore \mathrm{NT} = \frac{\mathrm{PF_1} \times \mathrm{PF_2}}{\mathrm{OQ}}$$

$$\therefore \mathrm{OQ} \times \mathrm{NT} = \mathrm{PF_1} \times \mathrm{PF_2} \qquad 4.4$$

Eqn. (4.4) is the mathematical expression of the theorem.

Theorem- 5

If points F_1 & F_2 are foci of the ellipse and PF_1 & PF_2 are pair focal distances, RS is sub-tangent and RM is sub-normal on conjugate axis through tangent point P, then the product of PF_1 & PF_2 is equal to product of ordinate of point P and the sum of sub tangent and sub-normal.

Proof for the theorem- 5

Referring the fig.3, a tangent is drawn at point P of an ellipse. F_1 & F_2 are the foci of the ellipse. a & b are the semi-major axis & semi-minor axis respectively, point O is centre of the ellipse. points T & S are the point of intersection of tangent with transverse axis & conjugate axis respectively, points Q & R are the projection of point P on transverse axis & conjugate axis. QT & QN are sub-tangent & sub-normal with respect to transverse axis, RS & RM are sub-tangent & sub-normal with respect to conjugate axis, OQ & OR are the abscissa & ordinate of point tangent point P on the ellipse respectively. OT & OS are x-intercept and y-intercept of a tangent drawn at point P for an ellipse respectively, ON & OM are x-intercept and y-intercept of a normal drawn at point P respectively.

$$\text{Eqn.}\,(12.4) \Longrightarrow \mathrm{OM} = \frac{(a^2 - b^2)\sin(\theta^\circ)}{b} \qquad 5.1$$

In fig.3, $\mathrm{MS} = \mathrm{OS} + \mathrm{OM}$

Substituting eqns. (3.9) and (3.19) in above

$$\therefore \mathrm{MS} = \left(\frac{b}{\sin(\theta^\circ)}\right) + \left(\frac{(a^2 - b^2)\sin(\theta^\circ)}{b}\right)$$

Simplifying the above eqn.,

$$\therefore \mathrm{MS} = \frac{a^2\sin^2(\theta^\circ) + b^2\cos^2(\theta^\circ)}{b\sin(\theta^\circ)} \qquad 5.2$$

Substituting, $b\sin(\theta^\circ) = \mathrm{OR}$ & substituting eqn. (3.17) in above eqn.,

$$\mathrm{MS} = \frac{\mathrm{PF_1} \times \mathrm{PF_2}}{\mathrm{OR}}$$

Simplifying the above eqn.,

$$\therefore \mathrm{OR} \times \mathrm{MS} = \mathrm{PF_1} \times \mathrm{PF_2} \qquad 5.3$$

Eqn. (5.3) is the mathematical expression of the theorem.

Theorem- 6:

If sub-normal is drawn to an ellipse through tangent point P and this normal intersect the conjugate axis at point M, points F_1 & F_2 are foci of the ellipse and PF_1 & PF_2 are the pair focal distances, then the ratio of square of PM and product of these pair focal distances is equal to the ratio of square of semi-major axis and square of semi-minor axis.

$$\frac{PM^2}{PF_1 \times PF_2} = \frac{a^2}{b^2}$$

Proof of the theorem- 6

Referring fig.3, a tangent is drawn at point P of an ellipse. F_1 & F_2 are the foci of the ellipse. a & b are the semi-major axis & semi-minor axis respectively, point O is centre of the ellipse. points T & S are the point of intersection of tangent with transverse axis & conjugate axis respectively, points Q & R are the projection of point P on transverse axis & conjugate axis. QT & QN are sub-tangent & sub-normal with respect to transverse axis, RS & RM are sub-tangent & sub-normal with respect to conjugate axis, OQ & OR are the abscissa & ordinate of point tangent point P on the ellipse respectively. OT & OS are x-intercept and y-intercept of a tangent drawn at point P for an ellipse respectively, ON & OM are x-intercept and y-intercept of a normal drawn at point P respectively.

$$\text{Eqn.}\,(3.21) \Longrightarrow \text{PM} = \frac{a\sqrt{a^2\sin^2(\theta^\circ) + b^2\cos^2(\theta^\circ)}}{b} \qquad 6.1$$

$$\therefore \text{PM}^2 = \frac{a^2}{b^2}[a^2\sin^2(\theta^\circ) + b^2\cos^2(\theta^\circ)]$$

Substituting eqn. (3.17) in above eqn.,

$$\therefore \text{PM}^2 = \frac{a^2}{b^2}(\text{PF}_1 \times \text{PF}_2)$$

$$\therefore \frac{\text{PM}^2}{\text{PF}_1 \times \text{PF}_2} = \frac{a^2}{b^2} \qquad 6.4$$

where, $\frac{a^2}{b^2}$ is an arbitrary constant.

Eqn. (6.4) is mathematical expression of the theorem.

Theorem- 7:

If sub-normal is drawn to an ellipse through tangent point P and this normal intersect the transverse axis at point N, points F_1 & F_2 are foci of the ellipse and PF_1 & PF_2 are the pair focal distances, then the ratio of square of PN and product of these pair focal distances is equal to the ratio of square of semi-minor axis and square of semi-major axis. This can be written in mathematical form as

$$\frac{PN^2}{PF_1 \times PF_2} = \frac{b^2}{a^2}$$

Proof of the theorem- 7

Referring fig.3, a tangent is drawn at point P of an ellipse. F_1 & F_2 are the foci of the ellipse. a & b are the semi-major axis & semi-minor axis respectively, point O is centre of the ellipse. points T & S are the point of intersection of tangent with transverse axis & conjugate axis respectively, points Q & R are the projection of point P on transverse axis & conjugate axis. QT & QN are sub-tangent & sub-normal with respect to transverse axis, RS & RM are sub-tangent & sub-normal with respect to conjugate axis, OQ & OR are the abscissa & ordinate of point tangent point P on the ellipse respectively. OT & OS are x-intercept and y-intercept of a tangent drawn at point P for an ellipse respectively, ON & OM are x-intercept and y-intercept of a normal drawn at point P respectively.

$$\text{Eqn.}\,(3.15) \Longrightarrow \text{PN} = \frac{b\sqrt{a^2\sin^2(\theta^\circ) + b^2\cos^2(\theta^\circ)}}{a}$$

$$\therefore \text{PN}^2 = \frac{b^2[a^2\sin^2(\theta^\circ) + b^2\cos^2(\theta^\circ)]}{a^2} \qquad 7.1$$

Substituting eqn. (3.17) in above eqn.,

$$\text{PN}^2 = \left(\frac{b}{a}\right)^2 \times \text{PF}_1 \times \text{PF}_2$$

$$\therefore \frac{\text{PN}^2}{\text{PF}_1 \times \text{PF}_2} = \frac{b^2}{a^2} \qquad 7.2$$

Eqn. (7.2) is mathematical expression of the theorem.

Theorem- 8

If sub-normal is drawn to an ellipse through tangent point P and this normal intersect the transverse axis at point N, points $F_1 \& F_2$ are foci of the ellipse and $PF_1 \& PF_2$ are the pair focal distances, then the ratio of product of PN & PT and product of these pair focal distances is equal to the ratio of ordinate to abscissa of the point P. This can be written in mathematical form as

$$\frac{PN \times PT}{PF_1 \times PF_2} = \frac{OR}{OQ}$$

Proof of the theorem- 8

Referring fig.3, a tangent is drawn at point P of an ellipse. $F_1 \& F_2$ are the foci of the ellipse. a & b are the semi-major axis & semi-minor axis respectively, point O is centre of the ellipse. points T & S are the point of intersection of tangent with transverse axis & conjugate axis respectively, points Q & R are the projection of point P on transverse axis & conjugate axis. QT & QN are sub-tangent & sub-normal with respect to transverse axis, RS & RM are sub-tangent & sub-normal with respect to conjugate axis, OQ & OR are the abscissa & ordinate of point tangent point P on the ellipse respectively. OT & OS are x-intercept and y-intercept of a tangent drawn at point P for an ellipse respectively, ON & OM are x-intercept and y-intercept of a normal drawn at point P respectively.

$$\text{Eqn.}(3.18) \Longrightarrow \mathrm{PN} \times \mathrm{PT} = \frac{[\mathrm{PF_1} \times \mathrm{PF_2}] \times \mathrm{bsin}(\theta^\circ)}{\mathrm{acos}(\theta^\circ)}$$

$$\therefore \frac{\mathrm{PN} \times \mathrm{PT}}{\mathrm{PF_1} \times \mathrm{PF_2}} = \frac{\mathrm{bsin}(\theta^\circ)}{\mathrm{acos}(\theta^\circ)}$$

Substituting, $\mathrm{bsin}(\theta^\circ) = \mathrm{PQ} = \mathrm{OR}$ and $\mathrm{acos}(\theta^\circ) = \mathrm{OQ}$ in above,

$$\frac{\mathrm{PN} \times \mathrm{PT}}{\mathrm{PF_1} \times \mathrm{PF_2}} = \frac{\mathrm{OR}}{\mathrm{OQ}} \qquad 8.1$$

Eqn. (8.1) is mathematical expression of the theorem.

Theorem- 9

If sub-normal is drawn to an ellipse through tangent point P and this normal intersect the auxiliary axis at point M, points $F_1 \& F_2$ are foci of the ellipse and $PF_1 \& PF_2$ are the pair focal distances, then the ratio of product of PM & PS and product of these pair focal distances is equal to the ratio of abscissa and ordinate of the point P. This can be written in mathematical form as

$$\frac{PM \times PS}{PF_1 \times PF_2} = \frac{OQ}{OR}$$

Proof of the theorem- 9

Referring fig.3, a tangent is drawn at point P of an ellipse. $F_1 \& F_2$ are the foci of the ellipse. a & b are the semi-major axis & semi-minor axis respectively, point O is centre of the ellipse. points T & S are the point of intersection of tangent with transverse axis & conjugate axis respectively, points Q & R are the projection of point P on transverse axis & conjugate axis. QT & QN are sub-tangent & sub-normal with respect to transverse axis, RS & RM are sub-tangent & sub-normal with respect to conjugate axis, OQ & OR are the abscissa & ordinate of point tangent point P on the ellipse respectively. OT & OS are x-intercept and y-intercept of a tangent drawn at point P for an ellipse respectively, ON & OM are x-intercept and y-intercept of a normal drawn at point P respectively.

$$\text{Eqn.}(3.23) \Longrightarrow \mathrm{PM} \times \mathrm{PS} = \frac{[\mathrm{PF_1} \times \mathrm{PF_2}] \times \mathrm{acos}(\theta^\circ)}{\mathrm{bsin}(\theta^\circ)}$$

$$\therefore \frac{\mathrm{PM} \times \mathrm{PS}}{\mathrm{PF_1} \times \mathrm{PF_2}} = \frac{\mathrm{acos}(\theta^\circ)}{\mathrm{bsin}(\theta^\circ)}$$

Substituting, $\mathrm{acos}(\theta^\circ) = \mathrm{OQ}$ and $\mathrm{bsin}(\theta^\circ) = \mathrm{PQ} = \mathrm{OR}$ in above,

$$\therefore \frac{\mathrm{PM} \times \mathrm{PS}}{\mathrm{PF_1} \times \mathrm{PF_2}} = \frac{\mathrm{OQ}}{\mathrm{OR}} \qquad 9.1$$

Eqn. (9.1) is mathematical expression of the theorem.

Theorem- 10:

If points F_1 & F_2 are foci of an ellipse and PF_1 & PF_2 are the pair focal distances of point P, point P is tangent point on ellipse, points T & N are intersection of tangent and normal with transverse axis respectively and OQ is abscissa of the tangent point P, then

$$\frac{PT^2 + PN^2}{(PF_1 \times PF_2)^2} = \frac{1}{OQ^2}$$

Proof of the theorem- 10

Referring fig.3, a tangent is drawn at point P of an ellipse. F_1 & F_2 are the foci of the ellipse. a & b are the semi-major axis & semi-minor axis respectively, point O is centre of the ellipse. points T & S are the point of intersection of tangent with transverse axis & conjugate axis respectively, points Q & R are the projection of point P on transverse axis & conjugate axis. QT & QN are sub-tangent & sub-normal with respect to transverse axis, RS & RM are sub-tangent & sub-normal with respect to conjugate axis, OQ & OR are the abscissa & ordinate of point tangent point P on the ellipse respectively. OT & OS are x-intercept and y-intercept of a tangent drawn at point P for an ellipse respectively, ON & OM are x-intercept and y-intercept of a normal drawn at point P respectively.

$$\text{Eqn.}(3.12) \Rightarrow \mathrm{PT} = \tan(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2\cos^2(\theta°)}$$

$$\therefore \mathrm{PT}^2 = \tan^2(\theta°) \times [a^2\sin^2(\theta°) + b^2\cos^2(\theta°)] \qquad 10.1$$

$$\text{Eqn.}(3.15) \Rightarrow \mathrm{PN} = \frac{b\sqrt{a^2\sin^2(\theta°) + b^2\cos^2(\theta°)}}{a}$$

$$\therefore \mathrm{PN}^2 = \frac{b^2 \times [a^2\sin^2(\theta°) + b^2\cos^2(\theta°)]}{a^2} \qquad ---------------------(10.2)$$

Adding eqns. (10.1) and (10.2)

$$\mathrm{PT}^2 + \mathrm{PN}^2 = \left(\frac{\sin^2(\theta°) \times [a^2\sin^2(\theta°) + b^2\cos^2(\theta°)]}{\cos^2(\theta°)}\right) + \left(\frac{b^2 \times [a^2\sin^2(\theta°) + b^2\cos^2(\theta°)]}{a^2}\right)$$

Substituting eqn. (3.17) in above eqn.,

$$\therefore \mathrm{PT}^2 + \mathrm{PN}^2 = \left(\frac{a^2\sin^2(\theta°) \times [\mathrm{PF_1} \times \mathrm{PF_2}] + b^2\cos^2(\theta°) \times [\mathrm{PF_1} \times \mathrm{PF_2}]}{a^2\cos^2(\theta°)}\right)$$

Simplifying the above eqn.,

$$\therefore \mathrm{PT}^2 + \mathrm{PN}^2 = \left(\frac{[\mathrm{PF_1} \times \mathrm{PF_2}]^2}{a^2\cos^2(\theta°)}\right)$$

Substituting, $a\cos(\theta°) = \mathrm{OQ}$ in above,

$$\therefore \mathrm{PT}^2 + \mathrm{PN}^2 = \frac{(\mathrm{PF_1} \times \mathrm{PF_2})^2}{\mathrm{OQ}^2}$$

$$\therefore \frac{\mathrm{PT}^2 + \mathrm{PN}^2}{(\mathrm{PF_1} \times \mathrm{PF_2})^2} = \frac{1}{\mathrm{OQ}^2} \qquad 10.2$$

Eqn. (10.3) is mathematical expression of the theorem.

Theorem- 11

If points F_1 & F_2 are foci of an ellipse and PF_1 & PF_2 are the pair focal distances of point P, point P is tangent point on ellipse, points S & M are intersection of tangent and normal with transverse axis respectively and OR is ordinate of the tangent point P, then

$$\frac{PS^2 + PM^2}{(PF_1 \times PF_2)^2} = \frac{1}{OR^2}$$

Proof of the theorem- 11

Referring fig.3, a tangent is drawn at point P of an ellipse. F_1 & F_2 are the foci of the ellipse. a & b are the semi-major axis & semi-minor axis respectively, point O is centre of the ellipse. points T & S are the point of intersection of tangent with transverse axis & conjugate axis respectively, points Q & R are the projection of point P on transverse axis & conjugate axis. QT & QN are sub-tangent & sub-normal with respect to transverse axis, RS & RM are sub-tangent & sub-normal with respect to conjugate axis, OQ & OR are the abscissa & ordinate of point tangent point P on the ellipse respectively. OT & OS are x-intercept and y-intercept of a tangent drawn at point P for an ellipse respectively, ON & OM are x-intercept and y-intercept of a normal drawn at point P respectively.

$$\text{Eqn.}\,(3.22) \Longrightarrow \mathrm{PS} = \frac{\sqrt{a^2\sin^2(\theta°) + b^2\cos^2(\theta°)}}{\tan(\theta°)}$$

$$\therefore \mathrm{PS}^2 = \frac{a^2\sin^2(\theta°) + b^2\cos^2(\theta°)}{\tan^2(\theta°)} \qquad 11.1$$

$$\text{Eqn.}\,(3.21) \Longrightarrow \mathrm{PM} = \frac{a\sqrt{a^2\sin^2(\theta°) + b^2\cos^2(\theta°)}}{b}$$

$$\therefore \mathrm{PM}^2 = \frac{a^2[a^2\sin^2(\theta°) + b^2\cos^2(\theta°)]}{b^2} \qquad 11.2$$

Adding eqns. (11.1) & (11.2),

$$\mathrm{PS}^2 + \mathrm{PM}^2 = \left(\frac{a^2\sin^2(\theta°) + b^2\cos^2(\theta°)}{\tan^2(\theta°)}\right) + \left(\frac{a^2[a^2\sin^2(\theta°) + b^2\cos^2(\theta°)]}{b^2}\right)$$

Substituting eqn. (3.17) in above eqn.,

$$\mathrm{PS}^2 + \mathrm{PM}^2 = \left(\frac{\mathrm{PF_1} \times \mathrm{PF_2}}{\tan^2(\theta°)}\right) + \left(\frac{a^2[\mathrm{PF_1} \times \mathrm{PF_2}]}{b^2}\right)$$

Simplifying the above eqn.,

$$\therefore \mathrm{PS}^2 + \mathrm{PM}^2 = \frac{(\mathrm{PF_1} \times \mathrm{PF_2})^2}{b^2\sin^2(\theta°)}$$

Substituting, $b\sin(\theta°) = \mathrm{OR}$ in above,

$$\therefore \mathrm{PS}^2 + \mathrm{PM}^2 = \frac{(\mathrm{PF_1} \times \mathrm{PF_2})^2}{\mathrm{OR}^2}$$

$$\therefore \frac{\mathrm{PS}^2 + \mathrm{PM}^2}{(\mathrm{PF_1} \times \mathrm{PF_2})^2} = \frac{1}{\mathrm{OR}^2} \qquad 11.3$$

Eqn. (11.3) is mathematical expression of the theorem.

Theorem- 12

If points F_1 & F_2 are foci of an ellipse and PF_1 & PF_2 are the pair focal distances of point P, point P is tangent point on ellipse, points T & S are intersection of tangent with transverse axis & conjugate axis respectively, similarly points N & M are intersection of normal with transverse axis & auxiliary axis respectively, then

$$PF_1 \times PF_2 = \left(\frac{OQ \times OR}{OP}\right) \times \sqrt{PT^2 + PN^2 + PS^2 + PM^2}$$

Proof of the theorem- 12

Referring fig.3, a tangent is drawn at point P of an ellipse. $F_1 \& F_2$ are the foci of the ellipse. a & b are the semi-major axis & semi-minor axis respectively, point O is centre of the ellipse. points T & S are the point of intersection of tangent with transverse axis & conjugate axis respectively, points Q & R are the projection of point P on transverse axis & conjugate axis. QT & QN are sub-tangent & sub-normal with respect to transverse axis, RS & RM are sub-tangent & sub-normal with respect to conjugate axis, OQ & OR are the abscissa & ordinate of point tangent point P on the ellipse respectively. OT & OS are x-intercept and y-intercept of a tangent drawn at point P for an ellipse respectively, ON & OM are x-intercept and y-intercept of a normal drawn at point P respectively.

$$\text{Eqn.}(10.3) \Longrightarrow \frac{PT^2+PN^2}{(PF_1\times PF_2)^2}=\frac{1}{OQ^2} \tag{12.1}$$

$$\text{Eqn.}(11.3) \Longrightarrow \frac{PS^2+PM^2}{(PF_1\times PF_2)^2}=\frac{1}{OR^2} \tag{12.2}$$

Adding eqns. (12.1) and (12.2),

$$\left(\frac{PT^2+PN^2}{(PF_1\times PF_2)^2}\right)+\left(\frac{PS^2+PM^2}{(PF_1\times PF_2)^2}\right)=\frac{1}{OQ^2}+\frac{1}{OR^2}$$

$$\left(\frac{PT^2+PN^2}{(PF_1\times PF_2)^2}\right)+\left(\frac{PS^2+PM^2}{(PF_1\times PF_2)^2}\right)=\frac{(OR^2+OQ^2)}{OQ^2\times OR^2}$$

Substituting eqn. (14.13) in above,

$$\left(\frac{PT^2+PN^2}{(PF_1\times PF_2)^2}\right)+\left(\frac{PS^2+PM^2}{(PF_1\times PF_2)^2}\right)=\frac{OP^2}{OQ^2\times OR^2}$$

Simplifying the above eqn.,

$$\therefore \frac{PT^2+PN^2+PS^2+PM^2}{(PF_1\times PF_2)^2}=\left(\frac{OP}{OQ\times OR}\right)^2 \tag{12.3}$$

Eqn. (12.3) is mathematical expression of the theorem.

Theorem- 13

If a tangent is drawn at point P for an ellipse, OQ & OR are parametric co-ordinates of point P, points T & S are intersection of tangent with transverse axis & conjugate axis respectively, similarly points N & M are intersection of normal with transverse axis & conjugate axis respectively, OP & OJ are semi-diameter & semi-conjugate diameter, points F_1 & F_2 are foci of the ellipse, then

$$(QN+QT)\times(RM+RS)=\frac{(PF_1\times PF_2)^2}{OQ\times OR}$$

Proof of the theorem- 13

Referring fig.3, a tangent is drawn at point P of an ellipse. $F_1 \& F_2$ are the foci of the ellipse. a & b are the semi-major axis & semi-minor axis respectively, point O is centre of the ellipse. points T & S are the point of intersection of tangent with transverse axis & conjugate axis respectively, points Q & R are the projection of point P on transverse axis & conjugate axis. QT & QN are sub-tangent & sub-normal with respect to transverse axis, RS & RM are sub-tangent & sub-normal with respect to conjugate axis, OQ & OR are the abscissa & ordinate of point tangent point P on the ellipse respectively. OT & OS are x-intercept and y-intercept of a tangent drawn at point P for an ellipse respectively, ON & OM are x-intercept and y-intercept of a normal drawn at point P respectively.

$$\text{Eqn.}(3.14) \Longrightarrow NQ=\frac{b^2\cos(\theta^\circ)}{a} \tag{13.1}$$

$$\text{Eqn.}\,(3.5) \implies QT = \frac{a\sin^2(\theta^\circ)}{\cos(\theta^\circ)} \quad 13.2$$

Adding eqns. (13.1) and (13.2),

$$QN + QT = \left(\frac{b^2\cos(\theta^\circ)}{a}\right) + \left(\frac{a\sin^2(\theta^\circ)}{\cos(\theta^\circ)}\right)$$

Simplifying the above eqn.,

$$QN + QT = \frac{a^2\sin^2(\theta^\circ) + b^2\cos^2(\theta^\circ)}{a\cos(\theta^\circ)} \quad 13.3$$

$$RM = OR + OM$$

Substituting eqns. (3.2) & (3.19) in above,

$$\therefore RM = b\sin(\theta^\circ) + \left(\frac{(a^2 - b^2)\sin(\theta^\circ)}{b}\right)$$

Simplifying the above eqn.,

$$RM = \frac{a^2\sin(\theta^\circ)}{b} \quad 13.4$$

$$\text{Eqn.}\,(3.8) \implies RS = \frac{b\cos^2(\theta^\circ)}{\sin(\theta^\circ)} \quad 13.5$$

Adding eqns. (13.4) and (13.5),

$$RM + RS = \left(\frac{a^2\sin(\theta^\circ)}{b}\right) + \left(\frac{b\cos^2(\theta^\circ)}{\sin(\theta^\circ)}\right)$$

Simplifying the above eqn.,

$$RM + RS = \frac{a^2\sin^2(\theta^\circ) + b^2\cos^2(\theta^\circ)}{b\sin(\theta^\circ)} \quad 13.6$$

Multiplying eqns. (13.3) & (13.6)

$$(NQ + QT) \times (RM + RS) = \left(\frac{a^2\sin^2(\theta^\circ) + b^2\cos^2(\theta^\circ)}{a\cos(\theta^\circ)}\right) \times \left(\frac{a^2\sin^2(\theta^\circ) + b^2\cos^2(\theta^\circ)}{b\sin(\theta^\circ)}\right)$$

Substituting $a\cos(\theta^\circ) = OQ$, $b\sin(\theta^\circ) = OR$ and substituting eqn. (3.17) in above eqn.,

$$(NQ + QT) \times (RM + RS) = \left(\frac{PF_1 \times PF_2}{OQ}\right) \times \left(\frac{PF_1 \times PF_2}{OR}\right)$$

$$\therefore (NQ + QT) \times (RM + RS) = \frac{(PF_1 \times PF_2)^2}{OQ \times OR} \quad 13.7$$

Eqn. (13.7) is mathematical expression of the theorem.

Theorem- 14

In an ellipse, if point T & point S are intersection of a tangent at point P with transverse axis & conjugate axis respectively, points N & M are intersection of normal with transverse axis & conjugate axis respectively, points Q & R are projection of point P on transverse axis & conjugate axis respectively, OQ & OR are parametric co-ordinates of point P, similarly QT & RS are sub-tangent on transverse axis & sub-tangent on conjugate axis respectively, points F_1 & F_2 are foci of the ellipse, OP & OJ are semi-diameter & semi-conjugate diameter, then

$$PF_1 \times PF_2 \equiv PT \times PS \equiv PN \times PM \equiv OJ \times OK \equiv OQ \times NT \equiv OR \times MS$$

$$\equiv BC^2 - OP^2 \equiv (TF_1 \times TF_2) - PT^2 \equiv (SF_1 \times SF_2) - PS^2$$

$$\equiv PN \times PT \times \left(\frac{OQ}{OR}\right) \equiv PM \times PS \times \left(\frac{OR}{OQ}\right) \equiv OQ\sqrt{PT^2 + PN^2} \equiv OR\sqrt{PS^2 + PM^2}$$

$$\equiv \sqrt{PT \times PN \times PS \times PM} \equiv \left(\frac{OQ \times OR}{OP}\right)\sqrt{PT^2 + PN^2 + PS^2 + PM^2}$$

$$\equiv \left(\frac{\overline{SF_1} \times \overline{OR}}{b}\right)^2 \equiv \overline{TF_1} \times \overline{TF_2} \times \left(\frac{\overline{OQ}}{a}\right)^2 \equiv \sqrt{\mathrm{OQ \times OR \times (NQ + QT) \times (RM + RS)}}$$

Proof of the theorem- 14
Referring fig.4,

(i) $\mathrm{PF_1 \times PF_2 = PT \times PS}$ $(Ara. Kalaimaran, 2025)$ 14.1

(ii) $\mathrm{PF_1 \times PF_2 = PN \times PM}$ $(Ara. Kalaimaran, 2025)$ 14.2

(iii) $\mathrm{PF_1 \times PF_2 = OJ \times OK = OJ^2}$ $(Ara. Kalaimaran, 2025)$ 14.3
Note: OJ & OK are equal

(iv) Eqn. $(4.4) \Longrightarrow \mathrm{PF_1 \times PF_1 = OQ \times NT}$ 14.4

(v) Eqn. $(5.3) \Longrightarrow \mathrm{PF_1 \times PF_1 = OR \times MS}$ 14.5

(vi) $\mathrm{PF_1 \times PF_2 = BC^2 - OP^2}$ $(Ara. Kalaimaran, 2025)$ 14.6

(vii) $\mathrm{PF_1 \times PF_2 = (TF_1 \times TF_2) - PT^2}$ $(Ara. Kalaimaran, 2025)$ 14.7

(viii) $\mathrm{PF_1 \times PF_2 = (SF_1 \times SF_2) - PS^2}$ $(Ara. Kalaimaran, 2025)$ 14.8

Note: $\mathrm{SF_1}$ & $\mathrm{SF_2}$ are equal due to symmetry about conjugate axis.

(ix) Eqn. $(8.4) \Longrightarrow \dfrac{\mathrm{PN \times PT}}{\mathrm{PF_1 \times PF_2}} = \dfrac{\mathrm{OR}}{\mathrm{OQ}}$

$$\therefore \mathrm{PF_1 \times PF_2 = PN \times PT \times \left(\frac{OQ}{OR}\right)} \quad 14.9$$

(x) Eqn. $(9.4) \Longrightarrow \dfrac{\mathrm{PM \times PS}}{\mathrm{PF_1 \times PF_2}} = \dfrac{\mathrm{OQ}}{\mathrm{OR}}$

$$\therefore \mathrm{PF_1 \times PF_2 = PM \times PS \times \left(\frac{OR}{OQ}\right)} \quad 14.10$$

(xi) Eqn. $(10.3) \Longrightarrow \dfrac{\mathrm{PT^2 + PN^2}}{\mathrm{(PF_1 \times PF_2)^2}} = \dfrac{1}{\mathrm{OQ^2}}$

$$\therefore \mathrm{PF_1 \times PF_2 = OQ\sqrt{PT^2 + PN^2}} \quad 14.11$$

(xii) Eqn. (11.3) $\Rightarrow \dfrac{PS^2 + PM^2}{(PF_1 \times PF_2)^2} = \dfrac{1}{OR^2}$

$\therefore PF_1 \times PF_2 = OR\sqrt{PS^2 + PM^2}$ 14.12

(xiii) Eqn. (14.3) $\Rightarrow PN \times PT \times PM \times PS = (PF_1 \times PF_2)^2$

$\therefore PF_1 \times PF_2 = \sqrt{PN \times PT \times PM \times PS}$ 14.13

(xiv) Eqn. (12.3) $\Rightarrow \dfrac{PT^2 + PN^2 + PS^2 + PM^2}{(PF_1 \times PF_2)^2} = \left(\dfrac{OP}{OQ \times OR}\right)^2$

$\therefore PF_1 \times PF_2 = \left(\dfrac{OQ \times OR}{OP}\right) \times \sqrt{PT^2 + PN^2 + PS^2 + PM^2}$ 14.14

(xv) Eqn. (13.7) $\Rightarrow (NQ + QT) \times (RM + RS) = \dfrac{(PF_1 \times PF_2)^2}{OQ \times OR}$

$\therefore PF_1 \times PF_2 = \sqrt{(NQ + QT) \times (RM + RS) \times OQ \times OR}$ 14.15

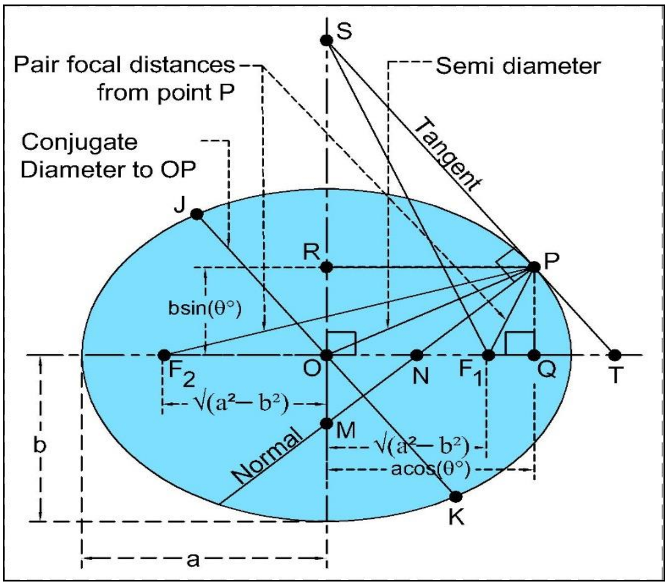

Fig 4

Compiling eqns. (14.1) to (14.15), it is concluded that the mathematical relation to each other parameters are expressed as

$$PF_1 \times PF_2 \equiv PT \times PS \equiv PN \times PM \equiv OJ \times OK \equiv OQ \times NT \equiv OR \times MS \equiv BC^2 - OP^2$$

$$\equiv (TF_1 \times TF_2) - PT^2 \equiv (SF_1 \times SF_2) - PS^2 \equiv PN \times PT \times \left(\frac{OQ}{OR}\right)$$

$$\equiv PM \times PS \times \left(\frac{OR}{OQ}\right) \equiv OQ\sqrt{PT^2 + PN^2} \equiv OR\sqrt{PS^2 + PM^2}$$

$$\equiv \sqrt{PT \times PN \times PS \times PM} \equiv \left(\frac{OQ \times OR}{OP}\right) \times \sqrt{PT^2 + PN^2 + PS^2 + PM^2}$$

$$\equiv \sqrt{OQ \times OR \times (QN + QT) \times (RM + RS)} \qquad ------------(14.16)$$

Eqn. (14.16) is the mathematical expression of the theorem.

Theorem- 15

If points F_1 & F_2 are foci of an ellipse and PF_1 & PF_2 are the pair focal distances of point P, point P is tangent point on ellipse, points N & M are intersection of normal with transverse axis & auxiliary axis respectively, then

$$\frac{(PN + PM)^2}{PF_1 \times PF_2} = \left(\frac{a^2 + b^2}{ab}\right)^2$$

Proof of the theorem- 15

Referring fig.4,

$$\text{Eqn.}(3.15) \Rightarrow \mathrm{PN} = \frac{b\sqrt{a^2\sin^2(\theta^\circ) + b^2\cos^2(\theta^\circ)}}{a} \quad 15.1$$

$$\text{Eqn.}(3.21) \Rightarrow \mathrm{PM} = \frac{a\sqrt{a^2\sin^2(\theta^\circ) + b^2\cos^2(\theta^\circ)}}{b} \quad 15.2$$

Adding eqns. (15.1) and (15.2),

$$\mathrm{PN} + \mathrm{PM} = \left(\frac{b\sqrt{a^2\sin^2(\theta^\circ) + b^2\cos^2(\theta^\circ)}}{a}\right) + \left(\frac{a\sqrt{a^2\sin^2(\theta^\circ) + b^2\cos^2(\theta^\circ)}}{b}\right)$$

Simplifying the above eqn.,

$$\therefore \mathrm{PN} + \mathrm{PM} = \frac{(a^2 + b^2)\sqrt{a^2\sin^2(\theta^\circ) + b^2\cos^2(\theta^\circ)}}{ab}$$

Substituting eqn. (3.17) in above eqn.,

$$\mathrm{PN} + \mathrm{PM} = \frac{(a^2 + b^2)\sqrt{\mathrm{PF_1} \times \mathrm{PF_2}}}{ab}$$

$$\therefore \frac{(\mathrm{PN} + \mathrm{PM})ab}{(a^2 + b^2)} = \sqrt{\mathrm{PF_1} \times \mathrm{PF_2}}$$

$$\therefore \frac{(\mathrm{PN} + \mathrm{PM})^2}{\mathrm{PF_1} \times \mathrm{PF_2}} = \left(\frac{a^2 + b^2}{ab}\right)^2 \quad 15.3$$

Eqn. (15.3) is mathematical expression of the theorem.

Theorem- 16

If tangent TS is drawn to an ellipse at a point P, points T and S are x-intercept & y-intercept of the tangent with transverse axis and conjugate axis respectively, points F_1 & F_2 are foci of the ellipse, OQ & OR are parametric co-ordinates of the point P, then

$$TS^2 = PF_1 \times PF_2 \times \left(\frac{ab}{OQ \times OR}\right)^2$$

Proof for the theorem- 16

Referring fig.4,

$$\text{Eqn. } (3.6) \Rightarrow \mathrm{OT} = \frac{a}{\cos(\theta^\circ)} \quad 16.1$$

$$\text{Eqn. } (3.9) \Rightarrow \mathrm{OS} = \frac{b}{\sin(\theta^\circ)} \quad 16.2$$

According to Pythagoras theorem on right Δ TOS,

$\mathrm{TS}^2 = \mathrm{OT}^2 + \mathrm{OS}^2$

Substituting eqns. (16.1) & (16.2) in above,

$$\mathrm{TS}^2 = \left(\frac{a}{\cos(\theta^\circ)}\right)^2 + \left(\frac{b}{\sin(\theta^\circ)}\right)^2$$

Simplifying the above eqn.,

$$\therefore \mathrm{TS}^2 = \frac{a^2\sin^2(\theta^\circ) + b^2\cos^2(\theta^\circ)}{\sin^2(\theta^\circ)\cos^2(\theta^\circ)} \qquad 16.3$$

Substituting eqn. (3.17) in above eqn.,

$$\therefore \mathrm{TS}^2 = \frac{\mathrm{PF}_1 \times \mathrm{PF}_2}{\sin^2(\theta^\circ)\cos^2(\theta^\circ)}$$

Simplifying the above eqn.,

$$\therefore \mathrm{TS}^2 = \mathrm{PF}_1 \times \mathrm{PF}_2 \times \left(\frac{ab}{\mathrm{OQ} \times \mathrm{OR}}\right)^2 \qquad 16.4$$

Eqn. (16.4) is the mathematical expression of the theorem.

Theorem- 17

If parametric co-ordinates of a point P on an ellipse are $[a\cos(\theta^\circ), b\sin(\theta^\circ)]$, points N & M are intersection of normal with transverse axis & conjugate axis respectively, points F_1 & F_2 are foci of the ellipse, then

$$\frac{MN \times PM}{PF_1 \times PF_2} = \frac{(a^2 - b^2)^2}{b^2}$$

Proof for the theorem- 17

Referring fig.4,

$$\text{Eqn.}(3.21) \Rightarrow \mathrm{PM} = \frac{a\sqrt{a^2\sin^2(\theta^\circ) + b^2\cos^2(\theta^\circ)}}{b} \qquad 17.1$$

$$\text{Eqn.}(3.20) \Rightarrow \mathrm{MN} = \frac{(a^2 - b^2)\sqrt{a^2\sin^2(\theta^\circ) + b^2\cos^2(\theta^\circ)}}{ab} \qquad 17.2$$

$$\mathrm{MN} \times \mathrm{PM} = \left(\frac{(a^2 - b^2)\sqrt{a^2\sin^2(\theta^\circ) + b^2\cos^2(\theta^\circ)}}{\cancel{a}b}\right) \times \left(\frac{\cancel{a}\sqrt{a^2\sin^2(\theta^\circ) + b^2\cos^2(\theta^\circ)}}{b}\right)$$

Substituting eqn. (3.17) in above eqn. & simplifying,

$$\therefore \frac{\mathrm{MN} \times \mathrm{PM}}{\mathrm{PF}_1 \times \mathrm{PF}_2} = \frac{(a^2 - b^2)}{b^2} \qquad 17.3$$

Eqn. (17.3) is the mathematical expression of the theorem.

Theorem- 18

If a tangent is drawn at point P for an ellipse, OR is ordinate of point P, point S is intersection of tangent with conjugate axis, points F_1 & F_2 are foci of the ellipse, then

$$\overline{PF_1} \times \overline{PF_2} = \left(\overline{SF_1}\right)^2 \times \left(\frac{\overline{OR}}{b}\right)^2$$

Proof of the theorem- 18

$$\text{Eqn. (5.2)} \Rightarrow OS = \frac{b}{\sin(\theta°)} \quad 18.4$$

Referring fig. 32. In right triangle F_1OS, according to Pythagoras theorem on right- triangle,

$$\overline{SF_1}^2 = \overline{OF_1}^2 + \overline{OS}^2$$

Substituting, $\overline{OF_1} = \sqrt{a^2 - b^2}$ and eqn. (18.1) in above,

$$\overline{SF_1}^2 = \left(\sqrt{a^2 - b^2}\right)^2 + \left(\frac{b}{\sin(\theta°)}\right)^2$$

Simplifying the above eqn.,

$$\therefore \overline{SF_1}^2 = \frac{a^2\sin^2(\theta°) + b^2\cos^2(\theta°)}{\sin^2(\theta°)}$$

Substituting eqn. (3.17) in above eqn.,

$$\therefore \overline{SF_1}^2 = \frac{\overline{PF_1} \times \overline{PF_2}}{\sin^2(\theta°)} \quad 18.2$$

Multiplying both numerator and denominator with b^2,

$$\therefore \overline{SF_1}^2 = \frac{\overline{PF_1} \times \overline{PF_2} \times b^2}{b^2 \times \sin^2(\theta°)}$$

Substituting $b\sin(\theta°) = OR$ in above,

$$\therefore \overline{SF_1}^2 = \frac{\overline{PF_1} \times \overline{PF_2} \times b^2}{OR^2}$$

$$\therefore \overline{PF_1} \times \overline{PF_2} = \left(\overline{SF_1}\right)^2 \times \left(\frac{\overline{OR}}{b}\right)^2 \quad 18.3$$

Eqn. (18.3) is mathematical expression of the theorem.

Theorem- 19

If a tangent is drawn at point PIf a tangent is drawn at point P for an ellipse, OQ & OR are parametric co-ordinates of point P, points T & S are intersection of tangent with transverse axis & conjugate axis respectively, points F_1 & F_2 are foci of the ellipse, then

$$\overline{PF_1} \times \overline{PF_2} = \overline{TF_1} \times \overline{TF_2} \times \left(\frac{\overline{OQ}}{b}\right)^2$$

Proof of the theorem- 19

$$\text{Eqn. (5.2)} \Rightarrow OT = \frac{a}{\cos(\theta°)} \quad 19.1$$

Referring fig.4

$$\overline{TF_1} \times \overline{TF_2} = \left(\overline{OT} - \overline{OF_1}\right) \times \left(\overline{OT} + \overline{OF_1}\right)$$

$$\therefore \overline{TF_1} \times \overline{TF_2} = \left(\overline{OT}^2 - \overline{OF_1}\right)^2$$

Substituting, $\overline{OF_1} = \sqrt{a^2 - b^2}$ and eqn. (32.3) in above,

$$\therefore \overline{TF_1} \times \overline{TF_2} = \left(\frac{a}{\cos(\theta°)}\right)^2 - \left(\sqrt{a^2 - b^2}\right)^2$$

Simplifying the above eqn.,

$$\therefore \overline{TF_1} \times \overline{TF_2} = \frac{a^2\sin^2(\theta°) + b^2\cos^2(\theta°)}{\cos^2(\theta°)}$$

Substituting eqn. (3.17) in above eqn.,

$$\therefore \overline{TF_1} \times \overline{TF_2} = \frac{\overline{PF_1} \times \overline{PF_2}}{\cos^2(\theta°)} \qquad 19.2$$

Multiplying both numerator and denominator with a^2,

$$\therefore \overline{TF_1} \times \overline{TF_2} = \frac{\overline{PF_1} \times \overline{PF_2} \times a^2}{a^2\cos^2(\theta°)}$$

Substituting $a\cos(\theta°) = OQ$ in above,

$$\therefore \overline{TF_1} \times \overline{TF_2} = \frac{\overline{PF_1} \times \overline{PF_2} \times a^2}{OQ^2}$$

$$\therefore \overline{PF_1} \times \overline{PF_2} = \overline{TF_1} \times \overline{TF_2} \times \left(\frac{\overline{OQ}}{a}\right)^2 \qquad 19.3$$

Eqn. (19.3) is mathematical expression of the theorem

Theorem- 20
If a tangent is drawn at point P for an ellipse, OQ & OR are parametric co-ordinates of point P, points T & S are intersection of tangent with transverse axis & conjugate axis respectively, points F_1 & F_2 are foci of the ellipse, then

$$\left(\frac{1}{\overline{\overline{SF_1}}}\right)^2 + \left(\frac{1}{\overline{\overline{TF_1}}}\right)^2 = \left(\frac{1}{\overline{\overline{PF_1} \times \overline{PF_2}}}\right)^2$$

Proof of the theorem- 20
Referring fig.4

$$\text{Reciprocal of eqn. (19.2)} = \frac{1}{\overline{SF_1}^2} = \frac{\sin^2(\theta°)}{\overline{\overline{PF_1} \times \overline{PF_2}}} \qquad 20.1$$

$$\text{Reciprocal of eqn. (20.2)} = \frac{1}{\overline{\overline{TF_1} \times \overline{TF_2}}} = \frac{\cos^2(\theta°)}{\overline{\overline{PF_1} \times \overline{PF_2}}} \qquad 20.2$$

Adding eqns. (20.1) & (20.2),

$$\frac{1}{\overline{SF_1}^2} + \frac{1}{\overline{\overline{TF_1} \times \overline{TF_2}}} = \frac{\sin^2(\theta°)}{\overline{\overline{PF_1} \times \overline{PF_2}}} + \frac{\cos^2(\theta°)}{\overline{\overline{PF_1} \times \overline{PF_2}}}$$

Simplifying the above eqn.,

$$\therefore \frac{1}{\overline{PF_1} \times \overline{PF_2}} = \frac{1}{\overline{SF_1}^2} + \frac{1}{\overline{TF_1} \times \overline{TF_2}} \qquad 20.3$$

Eqn. (20.3) is mathematical expression of the theorem.

Theorem- 21

If points F_1 & F_2 are foci of an ellipse and PF_1 & PF_2 are the pair focal distances of point P, a tangent TS is drawn at point P on ellipse, points T is intersection point of the tangent with transverse axis, OQ & OR are abscissa and ordinate of the point P respectively, a & b are semi-major axis and semi-minor axis respectively, then

$$PT^2 = \left(\frac{a \times OR}{b \times OQ}\right)^2 \times PF_1 \times PF_2$$

Proof of the theorem- 21

Referring fig.4

$$\text{Eqn. (3.12)} \Rightarrow PT = \tan(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2\cos^2(\theta°)} \qquad 21.1$$

$$\therefore PT^2 = \frac{\sin^2(\theta°)[a^2\sin^2(\theta°) + b^2\cos^2(\theta°)]}{\cos^2(\theta°)}$$

Multiplying both numerator and denominator by a^2b^2,

$$\therefore PT^2 = \frac{a^2b^2 \times [\sin^2(\theta°)] \times [a^2\sin^2(\theta°) + b^2\cos^2(\theta°)]}{a^2b^2[\cos^2(\theta°)]}$$

$$\therefore PT^2 = \frac{a^2[b^2\sin^2(\theta°)] \times [a^2\sin^2(\theta°) + b^2\cos^2(\theta°)]}{b^2 \times [a^2\cos^2(\theta°)]}$$

Substituting eqns. (3.2), (3.1) & substituting eqn. (3.17) in above eqn.,

$$PT^2 = \left(\frac{a \times OR}{b \times OQ}\right)^2 \times PF_1 \times PF_2 \qquad 21.2$$

Eqn. (21.2) is mathematical expression of the theorem.

Theorem- 22

If points F_1 & F_2 are foci of an ellipse and PF_1 & PF_2 are the pair focal distances of point P, a tangent TS is drawn at point P on ellipse, point S is intersection point of the tangent with conjugate axis, OQ & OR are abscissa and ordinate of the point P respectively, a & b are semi-major axis and semi-minor axis respectively, then

$$PS^2 = \left(\frac{b \times OQ}{a \times OR}\right)^2 \times PF_1 \times PF_2$$

Proof of the theorem- 22

$$\text{Eqn. (14.8)} \Rightarrow PS = \cot(\theta°)\sqrt{a^2\sin^2(\theta°) + b^2\cos^2(\theta°)} \qquad 22.1$$

$$\therefore PS^2 = \frac{\cos^2(\theta°)[a^2\sin^2(\theta°) + b^2\cos^2(\theta°)]}{\sin^2(\theta°)}$$

Multiplying both numerator and denominator by a^2b^2,

$$\therefore PS^2 = \frac{a^2b^2 \times \cos^2(\theta°) \times [a^2\sin^2(\theta°) + b^2\cos^2(\theta°)]}{a^2b^2 \times [\sin^2(\theta°)]}$$

$$\therefore \mathrm{PS}^2 = \frac{\mathrm{b}^2 \times [\mathrm{a}^2\cos^2(\theta°)] \times [\mathrm{a}^2\sin^2(\theta°) + \mathrm{b}^2\cos^2(\theta°)]}{\mathrm{a}^2 \times [\mathrm{b}^2\sin^2(\theta°)]}$$

Substituting eqns. (3.1), (3.2) & substituting eqn. (3.17) in above eqn.,

$$\mathrm{PS}^2 = \left(\frac{\mathrm{b} \times \mathrm{OQ}}{\mathrm{a} \times \mathrm{OR}}\right)^2 \times \mathrm{PF}_1 \times \mathrm{PF}_2 \qquad 22.2$$

Eqn. (22.2) is mathematical expression of the theorem.

Theorem- 23

In an ellipse, if point T & point S are intersection of a tangent at point P with transverse axis & conjugate axis respectively, points N & M are intersection of normal with transverse axis & conjugate axis respectively, points Q & R are projection of point P on transverse axis & conjugate axis respectively, OQ & OR are parametric co-ordinates of point P, similarly QT & RS are sub-tangent on transverse axis & sub-tangent on conjugate axis respectively, points F_1 & F_2 are foci of the ellipse, then

$$PF_1 \times PF_2 \equiv PN^2 \times \left(\frac{a^2}{b^2}\right) \equiv PM^2 \times \left(\frac{b^2}{a^2}\right) \equiv \frac{NF_1 \times NF_2}{e^2} \equiv \frac{MN \times PN}{e^2} \equiv PT^2 \times \left(\frac{b \times OQ}{a \times OR}\right)^2$$

$$\equiv PS^2 \times \left(\frac{a \times OR}{b \times OQ}\right)^2 \equiv MN \times PM \times \frac{b^2}{(a^2 - b^2)} \equiv MN^2 \times \left(\frac{ab}{a^2 - b^2}\right)^2 \equiv \left(\frac{\overline{SF_1} \times \overline{OR}}{b}\right)^2$$

$$\equiv \overline{TF_1} \times \overline{TF_2} \times \left(\frac{\overline{OQ}}{a}\right)^2 \equiv (PN + PM)^2 \times \left(\frac{ab}{a^2 + b^2}\right)^2 \equiv \left(\frac{TS \times OQ \times OR}{ab}\right)^2$$

Proof of the theorem- 23

(i) $\dfrac{\mathrm{PN}^2}{\mathrm{PF}_1 \times \mathrm{PF}_2} = \dfrac{\mathrm{b}^2}{\mathrm{a}^2}$ $(Ara.Kalaimaran, 2025)$

$$\therefore \mathrm{PF}_1 \times \mathrm{PF}_2 = \mathrm{PN}^2 \times \left(\frac{\mathrm{a}^2}{\mathrm{b}^2}\right) \qquad 23.1$$

(ii) Eqn. (6.4) $\Longrightarrow \dfrac{\mathrm{PM}^2}{\mathrm{PF}_1 \times \mathrm{PF}_2} = \dfrac{\mathrm{a}^2}{\mathrm{b}^2}$

$$\therefore \mathrm{PF}_1 \times \mathrm{PF}_2 = \mathrm{PM}^2 \times \left(\frac{\mathrm{b}^2}{\mathrm{a}^2}\right) \qquad 23.2$$

(iii) $\mathrm{PF}_1 = \dfrac{\mathrm{NF}_1 \times \mathrm{NF}_2}{\mathrm{PF}_1 \times \mathrm{PF}_2} = \dfrac{\mathrm{a}^2 - \mathrm{b}^2}{\mathrm{a}^2} = \mathrm{e}^2$ $(Ara.Kalaimaran, 2025)$

$$\therefore \mathrm{PF}_1 \times \mathrm{PF}_2 = \frac{\mathrm{NF}_1 \times \mathrm{NF}_2}{\mathrm{e}^2} \qquad 23.3$$

(iv) $\dfrac{\mathrm{MN} \times \mathrm{PN}}{\mathrm{PF}_1 \times \mathrm{PF}_2} = \mathrm{e}^2$ $(Ara.Kalaimaran, 2025)$

$$\therefore \mathrm{PF}_1 \times \mathrm{PF}_2 = \frac{\mathrm{MN} \times \mathrm{PN}}{\mathrm{e}^2} \qquad 23.4$$

(v) Eqn. (17.3) $\Longrightarrow \dfrac{\mathrm{MN} \times \mathrm{PM}}{\mathrm{PF}_1 \times \mathrm{PF}_2} = \dfrac{(\mathrm{a}^2 - \mathrm{b}^2)}{\mathrm{b}^2}$

$$\therefore \mathrm{PF_1 \times PF_2 = MN \times PM \times \frac{b^2}{(a^2-b^2)}} \quad 23.5$$

(vi) $\frac{\mathrm{MN}^2}{\mathrm{PF_1 \times PF_2}} = \left(\frac{a^2-b^2}{ab}\right)^2$ $(Ara.Kalaimaran, 2025)$

$$\therefore \mathrm{PF_1 \times PF_2 = MN^2} \times \left(\frac{ab}{a^2-b^2}\right)^2 \quad 23.6$$

(vii) Eqn. $(15.3) \Longrightarrow \frac{(\mathrm{PN+PM})^2}{\mathrm{PF_1 \times PF_2}} = \left(\frac{a^2+b^2}{ab}\right)^2$

$$\therefore \mathrm{PF_1 \times PF_2 = (PN+PM)^2} \left(\frac{ab}{a^2+b^2}\right)^2 \quad 23.7$$

(viii) Eqn. $(16.4) \Longrightarrow \mathrm{TS}^2 = \frac{\mathrm{PF_1 \times PF_2} \times a^2b^2}{\mathrm{OQ^2 \times OR^2}}$

$$\therefore \mathrm{PF_1 \times PF_2} = \left(\frac{\mathrm{TS \times OQ \times OR}}{ab}\right)^2 \quad 23.8$$

(ix) Eqn. $(18.3) \Longrightarrow \overline{\mathrm{PF_1}} \times \overline{\mathrm{PF_2}} = \left(\overline{\mathrm{SF_1}}\right)^2 \times \left(\frac{\overline{\mathrm{OR}}}{b}\right)^2$ 23.9

(x) Eqn. $(19.3) \Longrightarrow \overline{\mathrm{PF_1}} \times \overline{\mathrm{PF_2}} = \overline{\mathrm{TF_1}} \times \overline{\mathrm{TF_2}} \times \left(\frac{\overline{\mathrm{OQ}}}{a}\right)^2$ 23.10

(xi) Eqn. $(21.2) \Longrightarrow \mathrm{PT}^2 = \left(\frac{a \times \mathrm{OR}}{b \times \mathrm{OQ}}\right)^2 \times \mathrm{PF_1 \times PF_2}$

$$\therefore \mathrm{PF_1 \times PF_2 = PT^2} \times \left(\frac{b \times \mathrm{OQ}}{a \times \mathrm{OR}}\right)^2 \quad 23.11$$

(xii) Eqn. $(22.2) \Longrightarrow \left(\frac{b \times \mathrm{OQ}}{a \times \mathrm{OR}}\right)^2 \times \mathrm{PF_1 \times PF_2}$

$$\therefore \mathrm{PF_1 \times PF_2 = PS^2} \times \left(\frac{a \times \mathrm{OR}}{b \times \mathrm{OQ}}\right)^2 \quad 23.12$$

Compiling eqns. (23.1) to (23.12), it is concluded that the mathematical relation to each other parameters are expressed as:

$$\mathrm{PF_1 \times PF_2} \equiv \mathrm{PN}^2 \times \left(\frac{a^2}{b^2}\right) \equiv \mathrm{PM}^2 \times \left(\frac{b^2}{a^2}\right) \equiv \frac{\mathrm{NF_1 \times NF_2}}{e^2} \equiv \frac{\mathrm{MN \times PN}}{e^2} \equiv \mathrm{PT}^2 \times \left(\frac{b \times \mathrm{OQ}}{a \times \mathrm{OR}}\right)^2$$

$$\equiv \mathrm{PS}^2 \times \left(\frac{a \times \mathrm{OR}}{b \times \mathrm{OQ}}\right)^2 \equiv \mathrm{MN \times PM} \times \frac{b^2}{(a^2-b^2)} \equiv \mathrm{MN}^2 \times \left(\frac{ab}{a^2-b^2}\right)^2 \equiv \left(\frac{\overline{\mathrm{SF_1}} \times \overline{\mathrm{OR}}}{b}\right)^2$$

$$\equiv (\mathrm{PN+PM})^2 \times \left(\frac{ab}{a^2+b^2}\right)^2 \equiv \left(\frac{\mathrm{TS \times OQ \times OR}}{ab}\right)^2 \equiv \overline{\mathrm{TF_1}} \times \overline{\mathrm{TF_2}} \times \left(\frac{\overline{\mathrm{OQ}}}{a}\right)^2 \quad ---(23.13)$$

$$\text{where, } e = \sqrt{\frac{a^2 - b^2}{a^2}}$$

which is called as Linear eccentricity of ellipse (Borowski E.J & Borwein J.M, 1991) [1]
Eqn. (23.13) is mathematical expression of the theorem.

Theorem-24

If points F_1 & F_2 are foci of an ellipse and PF_1 & PF_2 are the pair focal distances of point P, a & b are semi-major axis & semi-minor axis respectively, $\angle QOP = \varphi°$, then

$$PF_1 \times PF_2 = \frac{a^4 \sin^2(\varphi°) + b^4 \cos^2 \varphi°}{a^2 \sin^2(\varphi°) + b^2 \cos^2(\varphi°)}$$

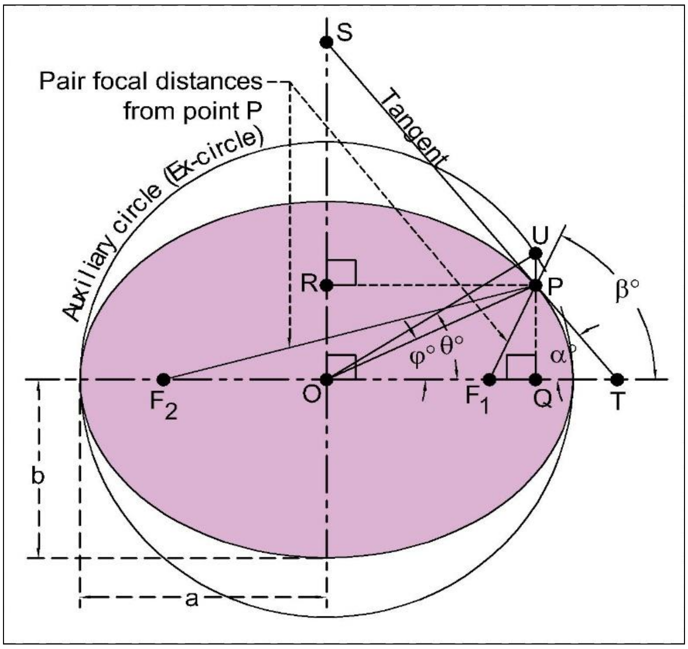

Fig 5

Proof for the theorem- 24

Referring fig. 5,

$$PF_1 = a - \left(\frac{b\sqrt{a^2 - b^2}\cos\varphi°}{\sqrt{a^2\sin^2(\varphi°) + b^2\cos^2(\varphi°)}}\right) (Ara.Kalaimaran, May, 2025) \tag{24.1}$$

$$PF_2 = a + \left(\frac{b\sqrt{a^2 - b^2}\cos\varphi°}{\sqrt{a^2\sin^2(\varphi°) + b^2\cos^2(\varphi°)}}\right) (Ara.Kalaimaran, May, 2025) \tag{24.2}$$

$$PF_1 \times PF_2 = \left[a - \left(\frac{b\sqrt{a^2 - b^2}\cos\varphi°}{\sqrt{a^2\sin^2(\varphi°) + b^2\cos^2(\varphi°)}}\right)\right] \times \left[a + \left(\frac{b\sqrt{a^2 - b^2}\cos\varphi°}{\sqrt{a^2\sin^2(\varphi°) + b^2\cos^2(\varphi°)}}\right)\right]$$

Simplifying the above eqn.,

$$\therefore PF_1 \times PF_2 = \frac{a^4\sin^2(\varphi°) + b^4\cos^2\varphi°}{a^2\sin^2(\varphi°) + b^2\cos^2(\varphi°)} \tag{24.3}$$

Eqn. (24.3) is the mathematical expression of the theorem.

Theorem- 25

If points F_1 & F_2 are foci of an ellipse and PF_1 & PF_2 are the pair focal distances of point P, point T is intersection of tangent at point P with transverse axis, a & b are semi-major axis & semi-minor axis respectively, $\angle QTP = \alpha°$, then

$$PF_1 \times PF_2 = \frac{a^2 b^2}{a^2 \sin^2(\alpha°) + b^2 \cos^2(\alpha°)}$$

Proof for the theorem- 25
Referring fig.5,

$$PF_1 = a\left(1 - \sqrt{\frac{(a^2 - b^2)\sin^2(\alpha°)}{a^2\sin^2(\alpha°) + b^2\cos^2(\alpha°)}}\right) (Ara.Kalaimaran, 2025) \quad 25.1$$

$$PF_2 = a\left(1 + \sqrt{\frac{(a^2 - b^2)\sin^2(\alpha°)}{a^2\sin^2(\alpha°) + b^2\cos^2(\alpha°)}}\right) (Ara.Kalaimaran, 2025) \quad 25.2$$

Multiplying eqns. (25.1) & (25.2),

$$PF_1 \times PF_2 = \left[a\left(1 - \sqrt{\frac{(a^2 - b^2)\sin^2(\alpha°)}{a^2\sin^2(\alpha°) + b^2\cos^2(\alpha°)}}\right)\right] \times \left[a\left(1 + \sqrt{\frac{(a^2 - b^2)\sin^2(\alpha°)}{a^2\sin^2(\alpha°) + b^2\cos^2(\alpha°)}}\right)\right]$$

Simplifying the above eqn.,

$$\therefore PF_1 \times PF_2 = \frac{a^2 b^2}{a^2\sin^2(\alpha°) + b^2\cos^2(\alpha°)} \quad 25.3$$

Eqn. (25.3) is the mathematical expression of the theorem.

Theorem- 26
If points F_1 & F_2 are foci of an ellipse and PF_1 & PF_2 are the pair focal distances of point P, a & b are semi-major axis & semi-minor axis respectively, $\angle QF_1P = \beta°$, then

$$PF_1 \times PF_2 = \frac{b^2\left[2a\left(a + \sqrt{a^2 - b^2}\cos\beta°\right) - b^2\right]}{\left(a + \sqrt{a^2 - b^2}\cos\beta°\right)^2}$$

Proof for the theorem- 26

$$\text{(xix)}\ \ PF_1 = \frac{b^2}{a + \sqrt{a^2 - b^2}\cos\beta°} \ \ (Ara.Kalaimaran, May, 2025) \quad 26.1$$

$$PF_2 = 2a - \left(\frac{b^2}{a + \sqrt{a^2 - b^2}\cos\beta°}\right) (Ara.Kalaimaran, May, 2025) \quad 26.2$$

Multiplying eqns. (26.1) and (26.2),

$$PF_1 \times PF_2 = \left(\frac{b^2}{a + \sqrt{a^2 - b^2}\cos\beta°}\right) \times \left[2a - \left(\frac{b^2}{a + \sqrt{a^2 - b^2}\cos\beta°}\right)\right]$$

Simplifying the above eqn.,

$$\therefore PF_1 \times PF_2 = \frac{b^2\left[2a\left(a + \sqrt{a^2 - b^2}\cos\beta°\right) - b^2\right]}{\left(a + \sqrt{a^2 - b^2}\cos\beta°\right)^2} \quad 26.3$$

Eqn. (26.3) is mathematical expression of the theorem.

Theorem- 27

If points F_1 & F_2 are foci of an ellipse and PF_1 & PF_2 are the pair focal distances of point P, a & b are semi-major axis & semi-minor axis respectively, $\angle UOQ = \theta°$, $\angle QTP = \alpha°$, $\angle QF_1P = \beta°$, $\angle POQ = \varphi°$, then

$$PF_1 \times PF_2 \equiv a^2\sin^2(\theta°) + b^2\cos^2(\theta°) \equiv \frac{a^2b^2}{a^2\sin^2(\alpha°)+b^2\cos^2(\alpha°)}$$

$$\equiv \frac{a^4\sin^2(\varphi°) + b^4\cos^2\varphi°}{a^2\sin^2(\varphi°) + b^2\cos^2(\varphi°)} \equiv \frac{b^2\left[2a\left(a+\sqrt{a^2-b^2}\cos\beta°\right)-b^2\right]}{\left(a+\sqrt{a^2-b^2}\cos\beta°\right)^2}$$

$$\equiv \left[2a - \left(\frac{b^2\left(a+\sqrt{a^2-b^2}\cos\delta°\right)}{a^2\sin^2(\delta°)+b^2\cos^2(\delta°)}\right)\right] \times \left[\frac{b^2\left(a+\sqrt{a^2-b^2}\cos\delta°\right)}{a^2\sin^2(\delta°)+b^2\cos^2(\delta°)}\right]$$

Proof of the theorem- 27

Referring fig. 27,

$$\text{(i) Eqn. (3.23)} \Longrightarrow PF_1 \times PF_2 = a^2\sin^2(\theta°) + b^2\cos^2\theta° \quad 27.1$$

$$\text{(ii) Eqn. (25.3)} \Longrightarrow PF_1 \times PF_2 = \frac{a^2b^2}{a^2\sin^2(\alpha°)+b^2\cos^2(\alpha°)} \quad 27.2$$

$$\text{(iii) Eqn. (24.1)} \Longrightarrow PF_1 \times PF_2 = \frac{a^4\sin^2(\varphi°)+b^4\cos^2\varphi°}{a^2\sin^2(\varphi°)+b^2\cos^2(\varphi°)} \quad 27.3$$

$$\text{(iii) Eqn. (26.3)} \Longrightarrow PF_1 \times PF_2 = \frac{b^2\left[2a\left(a+\sqrt{a^2-b^2}\cos\beta°\right)-b^2\right]}{\left(a+\sqrt{a^2-b^2}\cos\beta°\right)^2} \quad 27.4$$

Comparing eqns. (27.1) to (27.7), it is concluded that the mathematical relation to each other parameters are expressed as:

$$PF_1 \times PF_2 \equiv a^2\sin^2(\theta°) + b^2\cos^2(\theta°) \equiv \frac{a^2b^2}{a^2\sin^2(\alpha°)+b^2\cos^2(\alpha°)}$$

$$\equiv \frac{a^4\sin^2(\varphi°) + b^4\cos^2\varphi°}{a^2\sin^2(\varphi°) + b^2\cos^2(\varphi°)} \equiv \frac{b^2\left[2a\left(a+\sqrt{a^2-b^2}\cos\beta°\right)-b^2\right]}{\left(a+\sqrt{a^2-b^2}\cos\beta°\right)^2}$$

$$\equiv \left[2a - \left(\frac{b^2\left(a+\sqrt{a^2-b^2}\cos\delta°\right)}{a^2\sin^2(\delta°)+b^2\cos^2(\delta°)}\right)\right] \times \left[\frac{b^2\left(a+\sqrt{a^2-b^2}\cos\delta°\right)}{a^2\sin^2(\delta°)+b^2\cos^2(\delta°)}\right] \quad ------(27.5)$$

Eqn. (27.5) is mathematical expression of the theorem.

Conclusion

The author has developed and established 27 new theorems/ geometrical properties among various elements of an ellipse with necessary derivations and equations, proof of the theorems including appropriate drawings in detail. This article is described the mathematical relation of Product of pair focal distances of an ellipse with other elements of the ellipse. The mathematical expression of each theorem has also been given at beginning of the theorem. These theorems, which have been defined in this article will be very useful for those do research or further study in the field of conics & Euclidean geometry, also very useful as reference to the research scholars for higher-level research works, since these are also one of the important properties of an ellipse.

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