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On *rgwα*- regular space and *rgwα*- Normal space in topological spaces

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Abstract
The aim of this paper is to introduce $rgw\alpha$-τ_0,$rgw\alpha$-τ_1, $rgw\alpha$-τ_2,$rgw\alpha$-Regular space and $rgw\alpha$- Normal spaces and obtain their properties by utilising $rgw\alpha$-closed sets and also presented some characterizations of these spaces.

Keywords: $rgw\alpha$-closed sets,$rgw\alpha$-$\tau_0$space, $rgw\alpha$-$\tau_1$space, $rgw\alpha$-τ_2 space, $rgw\alpha$-Regular space and $rgw\alpha$- Normal space.

1. Introduction
Munshi [28] introduced g-regular and g-normal spaces using g-closed sets of Levine [47]. Maheshwari and Prasad [51] introduced the new class of spaces called s-normal spaces using semi-open sets. It was further studied by Noiri and Popa [73], Dorsett [2] and Arya [7]. Later, Benchalli *et al.* [18] and Shik John [79] studied the concept of g*-pre regular, g*-pre normal and w-normal, w-regular spaces in topological spaces. R. S. Wali and Vijayalaxmi R. Patil introduced and studied $rgw\alpha$- closed sets, RGWα- Continuous and RGWα- Irresolute maps in Topological spaces.
In this paper, new classes of spaces, namely $rgw\alpha$-τ_0,$rgw\alpha$-τ_1,$rgw\alpha$-τ_2,$rgw\alpha$-Regular space and $rgw\alpha$-Normal spaces were introduced and presented some characterizations of these spaces.

2. Priliminaries
Definition 2.1: A subset A of a space X is called a
1. α-open set [5] if A ⊆ int (cl(A)) and α-closed set of cl (int(cl(A)) ⊆ A.
2. Semi open set [7] if A ⊆ cl (int (A)) & semi closed set if int(cl(A)) ⊆A.
3. Regular open set [14] if A =int (cl (A)) and a regular closed set if A=cl (int(A)).
4. Regular α-open set [15] if there is a regular open set U ∋ U⊆A⊆ αcl(U)
5. Weakly closed set [13] (briefly, w-closed) if cl(A) ⊆U whenever A ⊆U & U is semi open in X.
6. Weakly α-closed set [2] (briefly, wα-closed) if αcl (A) ⊆U whenever A ⊆U & U is w-open in X.
7. Regular generalized weakly α-closed set [16] (briefly, $rgw\alpha$-closed) if if rαcl (A) ⊆U whenever A⊆U & U is weak α-open set in X.

Definition 2.2: A map f: (X, τ)→(Y, σ) is said to be
1. Regular-continuous(r-continuous) [1] if f-1(V) is r-closed in X for every closed subset V of Y.
2. Completely-continuous [1] if f-1(V) is regular closed in X for every closed subset V of Y.
3. Strongly-continuous [8] if f-1(V) is clopen (both open and closed) in X for every subset V of Y.
4. g-Continuous [8] if f-1(V) is g-closed in X for every closed subset V of Y
5. w-Continuous [12] if f-1(V) is w-closed in X for every closed subset V of Y

6. α-Continuous [9] if f-1(V) is α-closed in X for every closed subset V of Y.
7. wα-Continuous [2] if f-1(V) is wα-closed in X for every closed subset V of Y.
8. Irresolute [3] if f-1(V) is semi- closed in X for every semi-closed subset V of Y
9. α-Irresolute [5] if f-1(V) is α-closed in X for everyα-closed subset V of Y.

3. $rgw\alpha$-τ_0, $rgw\alpha$-τ_1, $rgw\alpha$-τ_2 Spaces and $rgw\alpha$-Regular Spaces

Definition 3.1: A topological space X is called argwα-τ_0 if for each pair of distinct points x, y of X, there exists a rgwα-open set G in X containing one of them and not the other.

Definition 3.2: A topological space X is called a rgwα-τ_1 if for each pair of distinct points x, y of X, there exists two rgwα-open sets G_1, G_2 in X such that $x \in G_1$, $y \notin G_1$, and $y \in G_2$, $x \notin G_2$.

Definition 3.3: A topological space X is called a rgwα-τ_2 (rgwα-Hausdorff) if for each pair of distinct points x, y of X there exists distinct rgwα-open sets H_1 and H_2 such that H_1 containing x but not y and H_2 containing y but not x.

Definition 3.4: A topological space X is said to be rgwα-regular if for each closed set F and a point $x \notin F$, there exist disjoint rgwα-open sets G and H such that $F \subseteq G$ and $x \in H$.

Example 3.5: Every discrete space is a $rgw\alpha$-regular space.
Proof: Let (X, D) be any discrete space. Let $x \in X$ and F be a closed set such that $x \notin F$, as every open set is $rgw\alpha$-open set in X,then{x} and F are $rgw\alpha$-open sets in X such that$x \in \{x\}$, $F \subset F$ and $\{x\} \cap F = \phi$. Thus, for all$x \in X$and each closed set F, $x \notin F$, there exist disjoint $rgw\alpha$-open sets. Hence X is $rgw\alpha$- regular space.

Theorem 3.6: Every regular space is rgwα-regular space.
Proof: Let X be a regular space. Let F be any closed set in X and $x \in X$ such that $x \notin F$. Since X is a regular space, there exists a pair of disjoint open sets G and H such that $F \subseteq G$ and $x \in H$. w.k.t.every open sets are rgwα-open, hence X is a rgwα-regular space.

Theorem 3.7: If X is a rgwα-regular space and is discrete space, then X isregular.
Proof: It can be proved as in discrete space every subset of X is both open and closed.

Theorem 3.8: Every α-regularisrgwα-regular space.
Proof: Let X be a α-regular space. Let F be any closed set in X and $x \in X$ such that $x \notin F$. By definition, there exists a pair of disjoint α-open sets G and H such that $F \subseteq G$ and $x \in H$. Hence X is a rgwα-regular space as everyα-open sets are rgwα-open sets.

Characterization 3.9: The following statements about the topological space are equivalent.
1. X is a $rgw\alpha$- regular space.
2. For each $x \in X$and each open setU containing x there exists an$rgw\alpha$-open set V $\ni x \in V \subset \bar{V} \subset U$.
3. For each $x \in X$ and each closed set A not containing x there exist an $rgw\alpha$- open sets V containing x such that $\bar{V} \cap A = \phi$.

Proof: (i)⟹(ii), Suppose (i) holds, i.e., Suppose X is a $rgw\alpha$-regular space, to prove (ii)

- Let $x \in X$ and U be an open set containing x, then $X - U$ is a closed set not containing x. Since X is$rgw\alpha$- regular there exists$rgw\alpha$-open sets V and W such that $x \in V, X - U \subset W$ and $V \cap W = \phi$.
- Now $V \cap W = \phi \Rightarrow V \subset X - W \Rightarrow \bar{V} \subset \overline{X - W} \Rightarrow \bar{V} \subset X - W$.
- Also $X - U \subset W \Longrightarrow X - W \subset X - [X - U] \Rightarrow X - W \subset U$.
- So $\bar{V} \subset X - W \subset U \Rightarrow \bar{V} \subset U$.
- Therefore, we have $x \in V \subset \bar{V} \subset U$. (ii)⟹(iii), Suppose(ii) holds to prove (iii).
- Let $x \in X$ and A be a closed set not containing x, then $X - A$is an open set containing x, therefore from the hypothesis (ii) there exist an $rgw\alpha$-open set V such that $x \in V \subset \bar{V} \subset X - A \Rightarrow x \in V, \bar{V} \subset X - A \Rightarrow x \in V, \bar{V} \cap A = \phi$ Thus for all $x \in X$ and for each closed set A not containing x there exist an $rgw\alpha$-open set V containing x such that $\bar{V} \cap A = \phi$. (iii)⟹(i), Suppose (iii) holds to prove (i), i.e., to prove X is $rgw\alpha$- regular.
- Let $x \in X$ and A be a closed set such that $x \notin A$ from (iii) there exist an $rgw\alpha$-open set V such that $x \in V$ and $\bar{V} \cap A = \phi \Rightarrow x \in V$ and $A \subset X - \bar{V}$. Note that V and $X - \bar{V}$ are disjoint $rgw\alpha$-open sets. Thus, for each $x \in X$ and each closed set A not containing x there exist $rgw\alpha$-open set V and $X - \bar{V}$ such that $x \in X, A \subset X - \bar{V}$ and $V \cap (X - \bar{V}) = \phi$. ⇒X is $rgw\alpha$-regular space.

Remark 3.10: $rgw\alpha$-regular space need not be $rgw\alpha$-τ_2 and need not be $rgw\alpha$ -τ_1 space.
Example 3.11: Consider X= {a, b, c} with topology τ={X, ϕ, {a}, {b,c}}, Here (X, τ) is $rgw\alpha$-regular space but not $rgw\alpha$-τ_2 and also not $rgw\alpha$ -τ_1 space.

Definition 3.12: A topological space X is said to be $rgw\alpha$ -τ_3 space if it is $rgw\alpha$- regular and $rgw\alpha$ -τ_1.

Theorem 3.13: Every $rgw\alpha$ -τ_3 space is $rgw\alpha$ -τ_2 space.

Proof: Let X be $rgw\alpha$ -τ_3 space then X is $rgw\alpha$- regular and $rgw\alpha$ -τ_1. To prove X is $rgw\alpha$ -τ_2. Let $x, y \in X$ with $x \neq y$. Now $x \neq y \Rightarrow x \notin \{y\}$, where {y} is closed set. Since X is $rgw\alpha$- regular space there exist $rgw\alpha$-open sets U and V such that $x \in U, \{y\} \subset V, U \cap V = \phi . \Rightarrow x \in U, y \in V, U \cap V = \phi$. Thus, X is $rgw\alpha$ -τ_2 space.

Remark 3.14

1) $rgw\alpha$-τ_3 $\Rightarrow$ $rgw\alpha$-τ_2 $\Rightarrow$ $rgw\alpha$-τ_1 $\Rightarrow$ $rgw\alpha$-τ_0
2) Converses are not true in general.

Theorem 3.15: Every subspace of $rgw\alpha$- regular space is $rgw\alpha$- regular.
Proof: Let (Y, τ_Y) be the subspace of (X, τ). $x \epsilon Y$ and F be a closed set in Y such that $x \notin F . \Rightarrow F = Y \cap A$, where A is closed in X. Also $x \notin F \Rightarrow x \notin A$. Further $x \in Y \subset X \Rightarrow x \in X$. Since X is $rgw\alpha$-regular, there exist $rgw\alpha$-open set U,V in X $\ni x \in U, A \subset V, U \cap V = \phi$. Now consider $Y \cap U$ and $Y \cap V$ clearly by definition τ_Y, $Y \cap U$ and $Y \cap V$ are $rgw\alpha$-open sets in Y. Further $x \in Y$ and $x \in U \Rightarrow x \in Y \cap U, A \subset V \Rightarrow, Y \cap A \subset Y \cap V \Rightarrow F \subset Y \cap V$ and $(Y \cap U) \cap (Y \cap V) = Y \cap U \cap V = Y \cap \phi = \phi$. Thus $\forall\, x \in Y$ and each closed set F in Y $\ni x \notin F$, there exist $rgw\alpha$-open sets in Y namely $Y \cap U$ and $Y \cap V \ni x \in Y \cap U, F \subset Y \cap V$ and $(Y \cap U) \cap (Y \cap V) = \phi$.

Theorem 3.16: $rgw\alpha$-regularity is a Topological property.
Proof: Let $f: X \to Y$ be a homeomorphism and X be a $rgw\alpha$-Regular space, to prove Y is also a $rgw\alpha$-Regular space. Let y$\in Y$ and F be a closed set in Y such that $y \notin F$. Since $y \in Y$ and $f: X \to Y$ is onto such that $f(x) = y$. Also we have $y \notin F \Rightarrow f(x) \notin F \Rightarrow f^{-'}(f(x)) \notin f^{-'}(F) \Rightarrow x \notin f^{-'}(F)$. Since f is continuous and F is closed in Y it follows that $f^{-'}(F)$ is closed in X. hence we have $x \in X, x \notin f^{-'}(F)$ is a closed set in X. Since X is $rgw\alpha$-Regular there exist $rgw\alpha$-open sets U,V in X such that $x \in U, f^{-'}(F) \subset V$ and $U \cap V = \phi$. It follows that f(U) and f(V) are $rgw\alpha$-open sets in Y. Also we have $x \in U \Rightarrow f(x) \in f(U) \Rightarrow y \in f(U)$. Also $f^{-'}(F) \subset V \Rightarrow f\left(f^{-'}(F)\right) \subset f(V) \Rightarrow F \subset f(V)$. Further consider $f(U) \cap f(V) = f(U \cap V) = f(\emptyset) = \emptyset$. Hence Y is $rgw\alpha$-regular space.

4. $rgw\alpha$-Normal Space and $rgw\alpha$-τ_4 Space

Definition 4.1: A topological space X is said to be $rgw\alpha$-normal if for each pair of disjoint closed sets, A,B in X there exist $rgw\alpha$-open sets U, V in X, such that $A \subset U, B \subset V$ and $U \cap V = \phi$.

- **Example 4.2:** Consider X={a, b, c} with topology τ={X, ϕ, {a}, {b}, {c}, {a, b},{a, c},{b, c}}, Here (X, τ) is $rgw\alpha$-Normal Space.
- **Example 4.3:** Every discrete space is $rgw\alpha$- Normal space.
- **Remark 4.4:** $rgw\alpha$- Normal space need not be $rgw\alpha$- Regular and need not be $rgw\alpha$-τ_1.
- **Example 4.5:** Consider X= {a, b, c} with topology τ={X, ϕ , {a}, {b}, {a,b}}, Here (X, τ) is $rgw\alpha$-normal space but not $rgw\alpha$-τ_2 and also not $rgw\alpha$ -τ_1 space.

Definition 4.6: A topological space X is said to be a $rgw\alpha$ τ_4-space if it is $rgw\alpha$-Normal and $rgw\alpha$-τ_1 space.
Theorem 4.7: Every $rgw\alpha$ -τ_4-space is $rgw\alpha$-τ_3 space.

Proof: Let X is $rgw\alpha$ -τ_4-space $\Rightarrow$ X is $rgw\alpha$-Normal and $rgw\alpha$-τ_1. $\Rightarrow$To prove X is $rgw\alpha$-regular. Let $x \in X$ and F be a closed set in X such that $x \notin F$. Since X is $rgw\alpha$-τ_1 space, {x} is $rgw\alpha$-closed set. Therefore, F and {x} are $rgw\alpha$-closed sets which are disjoint.
Since X is $rgw\alpha$-normal by definition their exist $rgw\alpha$-open sets U,V in X such that $F \subset U,\ \{x\} \subset V, U \cap V = \emptyset$.

Remark 4.8

1. $rgw\alpha$-τ_4 $\Rightarrow$ $rgw\alpha$-τ_3 $\Rightarrow$ $rgw\alpha$-τ_2 $\Rightarrow$ $rgw\alpha$-τ_1 $\Rightarrow$ $rgw\alpha$-τ_0.
2. The converses are not true in general.

Theorem 4.9: Every normal space is rgwα-normal.

- **Proof:** Let X be a normal space. Let A and B be a pair of disjoint closed sets in X. Then there exists a pair of disjoint open sets G and H in X such that A⊆G and B⊆H. Hence X is rgwα-normal as every closed sets are rgwα-closed sets.
- **Remark 4.10:** The converse need not be true in general.
- **Theorem 4.11**: If X is rgwα-normal and discrete space, then X is normal.

Proof: Let X be a rgwα-normal space. Let A and B be a pair of disjoint closed sets in X. Since X is normal, there exists a pair of disjoint open sets G and H in X such that A⊆G and B⊆H. Hence X is normal as every subset of X are both open and closed.

- **Theorem 4.12:** Every w-normal is rgwα-normal space.
- **Proof:** Let X be a w-normal space. Let A and B be a pair of disjoint closed sets in X. Then A and B are w-closed sets in X. Since X is w-normal, there exists a pair of disjoint open sets G and H in X such that A⊆G and B⊆H. Hence X is rgwα-normal as every open set is rgwα-open set.

Theorem 4.13: Characterization

The following statements about any topological space X are equivalent

1. X is $rgw\alpha$-normal space.
2. For each closed set A and each open set $U \ni A \subset U \ni$ an $rgw\alpha$-open set $U \ni A \subset V \subset \bar{V} \subset U$.
3. For each pair of disjoint closed sets, A, B in X an $rgw\alpha$-open set $U \ni A \subset U$ and $\bar{U} \cap B = \oint$.
4. For each pair of disjoint closed sets, A, B in X, there exist $rgw\alpha$-open sets U, V in X $\ni A \subset U, B \subset V$ and $\bar{U} \cap \bar{V} = \oint$.

Remark 4.14: $rgw\alpha$- Normality is not a hereditary property i.e. a subspace of a normal space need not be $rgw\alpha$-normal.

- **Theorem 4.15:** Every closed subspace of a $rgw\alpha$-normal space is $rgw\alpha$-normal.
- **Theorem 4.16:** $rgw\alpha$-Normality is invariant (preserved) under continuous closed surjections.
- **Corollary 4.17:** $rgw\alpha$-Normality is the topological property. i.e., $rgw\alpha$-Normality is invariant under a homeomorphism.
- **Corollary 4.18:** $rgw\alpha$-τ_4 property is invariant (preserved) under continuous closed surjections.
- **Corollary 4.19:** $rgw\alpha$-τ_4 property is the topological property. i.e., $rgw\alpha$-Normality is invariant under a homeomorphism.

5. Conclusion

In this paper we have introduced and studied the properties of rgwα-regular and rgwα-Normal spaces. Our future extension is rgwα- countability in Topological Spaces.

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7. Declaration: I, Dr. Vijayalaxmi Patil hereby declare that the research paper entitled "On $rgw\alpha$ Regular space and $rgw\alpha$-Normal space in Topological Spaces" submitted for publication is my original work and this paper has not been sent to anyone previously for any purpose.

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