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Nonlinear dynamics and chaos in fractional duffing system

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Abstract
Dynamical complexity of a Duffing oscillator in the presence of both a linear and fractional damping and excited harmonically has been investigated. The potential energy term involve in the system comprises two minima i.e., double well potential. Numerical simulation has been carried out with a view to understand the complex dynamics of such a fractional Duffing system with (*i*) increased values of the fractional order, α and (ii) increased value of the amplitude, F_0, of the external harmonic excitation while retaining the values of other parameters of the system. Time series, phase portraits have been extensively used to demonstrate pictorially the complex dynamics of the system. In addition, Lyapunov exponents have been computed in each of the cases (i) and (ii) to quantitatively asses the involved dynamical complexity.

Keywords: Thermoelectronics, semiconductor electronics, Stanislav Ordin

1. Introduction

A Duffing system has attracted the attention of several research in recent years [1]. Fractional calculus, though introduced almost three hundred years ago, introduces the possibility of differential and integral operators of arbitrary orders. In view of the observed complex frequency dependence of various damping materials, fractional calculus finds applications in many areas of engineering and physics [2, 3]. Fractional calculus has further been introduced earlier to understand the intricacies involved in controlling the dynamics of electronic oscillator such as Chua's circuit etc. [4]. Unlike the ordinary dynamical system requiring at least third order to exhibit chaotic dynamics, fractional order system is found to show chaotic dynamical pattern even when total fractional order is less than three [5, 6]. In the present work, we investigate the dynamical complexity of the Duffing system incorporating both linear and fractional damping terms. Using the various tools of nonlinear dynamics, both regular and chaotic behaviour of the system has been studied using phase space diagrams and Lyapunov characteristic exponent (LCE). The numerical simulation of the fractional Duffing system has been done when (i) fractional order of the system is varied and (ii) the amplitude of the sinusoidal/ harmonic term is varied.

In section 2, we briefly outline the mathematical formulation of the problem. The numerical simulation has been done in section 3. Results obtained are summarized in section 4.

2. Mathematical Formulation of the harmonically excited Fractional Duffing Equation

A Duffing oscillator with a linear damping term and when excited by a harmonic term may be written as:

$$\frac{d^2X}{dt^2} + \alpha\frac{dX}{dt} + \mu X + \rho X^3 = F_0 cos(\omega\, t) \tag{2.1}$$

Where, α is the coefficient of linear damping term, μ and ρ refers to the linear and cubic stiffness term respectively. Further, F_0 corresponds to the amplitude of the harmonic excitation with frequency, ω. The potential energy of the foregoing oscillator is given by

$$U(x) = \mu \frac{x^2}{2} + \rho \frac{x^4}{4} \quad (2.2)$$

For $\mu > 0, \rho > 0$, the potential is a single well type while for $\mu < 0, \rho > 0$, it is double-well type. Here after, we consider the potential function to be of the double-well type and the setting where $\mu = -1$ and $\rho = 1$.

In mechanical engineering applications, the involved damping is modelled using fractional order derivative [2] as it may describe complex dependence on frequency of the damping material. We therefore incorporate the fractional damping force, $F_{damping} = \beta \frac{d^q X}{dt^q}$, with $0 < q < 1$. Here q represents the fractional order. Therefore, eq. (2.1) could be rewritten as [2, 7]:

$$\frac{d^2X}{dt^2} + \alpha \frac{dX}{dt} + \beta \frac{d^qX}{dt^q} - X + X^3 = F_0 cos(\omega t) \quad (2.3)$$

Following the properties of fractional derivatives [8, 9], we observe that

$$D^q X(t) = D^{q_1} D^{q_2} \ldots . D^{q_{n-1}} D^{q_n} X(t) \quad (2.4)$$

Where,
$q = q_1 + q_2 \ldots \ldots \ldots + q_{n-1} + q_n$. Here we represent, $\frac{d^q X}{dt^q} = D^q X(t)$.

Using eq. (2.4), we may write eq.(2.3) as a set of following state equations [Litak, 2014].

$$D^1 X = Y;$$

$$D^{q_1} X = W; \quad (2.5)$$

$$D^1 Y = X(t) - X^3(t) - \beta W + F_0 \cos(\omega t)$$

In the literature on fractional differential equations [9], the mainly used form of the fractional order differential operator uses the form prescribed by Riemann-Liouville (RL), Caputo (C) and Grunwald-Letnikov (GL). The GL definition for a continuous function, $f(t)$, provides the following form for the fractional derivative:

$$\frac{d^q f}{dt^q} = D_t{}^q f(t) = \lim_{h \to 0} \frac{1}{h^q} \sum_{i=0}^{\infty} (-1)^i \binom{q}{i} f(t - ih) \quad (2.6)$$

Here, $\binom{q}{i}$ the binomial coefficients may be written in terms of Euler's Gamma function as:

$$\binom{q}{i} = C_i{}^q \frac{q!}{i!\,(q-i)!} = \frac{\Gamma(q+1)}{\Gamma(i+1)\Gamma(q-i+1)} \quad (2.7)$$

It may be noted that [8, 10],

$$C_0{}^q = 1; \; C_i{}^q = \left(1 - \frac{1+q}{i}\right) C_{i-1}{}^q.$$

On the other hand, RL and C definitions for the fractional derivatives may be written as follows:

The Riemann-Liouville definition of fractional derive is:

$${}^{RL}_{a}D_t{}^q f(t) = \frac{1}{\Gamma(n-q)} \frac{d^n}{dt^n} \int_a^t \frac{f(\tau)}{(t-\tau)^{q-n+1}} d\tau \quad (2.8)$$

Where,
$(n - 1 < q < n)$, a and t refers to the limits of operation ${}^{RL}_{a}D_t{}^q f(t)$.

The Caputo definition provides the following form for the fractional derivative:

$${}^{C}_{a}D_t{}^q f(t) = \frac{1}{\Gamma(n-q)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{q-n+1}} d\tau; \; (n - 1 < q < n) \quad (2.9)$$

In the present work, we follow the GL definition of the fractional derivative given in eq. (2.6).

For numerical solution of the fractional differential equation of the form

$$D_t{}^q Z(t) = f(Z(t)), \quad (2.10)$$

May be written as [9]:

$$Z(t_k) = f(Z(t_k), t_k) h^q - \sum_{i=1}^{k} C_i{}^q Z(t_{k-i}), \quad (2.11)$$

Where, the last term is called memory term, with $k = 1,2,3, \ldots . N$ and $N = \left[\frac{T_{sim}}{h}\right]$. Here T_{sim} denotes the total simulation time.

The generalized fractional order form of the Duffing system, eq. (2.5), may further be written in modified version as [3]:

$$D^{\nu_1} X = Y; \; D^{\nu_2} X = W;$$

$$D^{\nu_3} Y = X(t) - X^3(t) - \beta W + F_0 \cos(\omega U) \; ;$$

$$D^{\nu_4} U = 1. \quad (2.12)$$

Where, $\nu_i, i = 1,2,3,4$ are fractional orders and U refers to state of original time t. In the present work, we consider the case when $\nu_1 = \nu_3 = \nu_4 = 1$ and $\nu_2 = q_1$.

3. Numerical Simulation of the Fractional Duffing System

For numerical simulation work, we further rewrite eq.(2.12) into the following form:

$$\frac{dX}{dt} = Y;$$

$$\frac{dY}{dt} = -\alpha Y + X(t) - X^3(t) - \beta W + F_0 \cos(\omega U)$$

$$\frac{d^{q_1} X}{dt^{q_1}} = W; \quad (3.1)$$

$$\frac{dU}{dt} = 1.$$

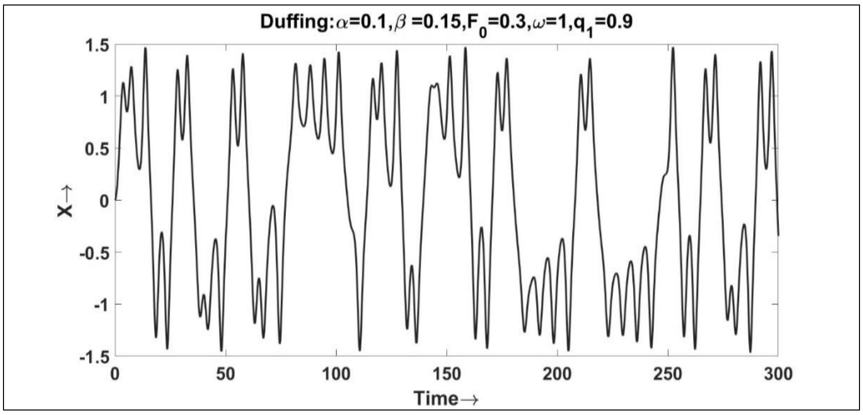
There are several control parameters such as α, β, F_0 and the fractional order q_1. In this paper, we consider the parameter $\alpha = 0.1$ and $\beta = 0.15$ in all the cases considered here. Further simulation has been carried out when (i) the fractional order q_1 varies keeping F_0 fixed and (ii) varying F_0 while keeping the fixed fractional order, q_1. The initial conditions were chosen as $\left[X, \dot{X}, \frac{d^{q_1} X}{dt^{q_1}}\right] = [0,0,0]$ at $t = 0$. In all the cases simulation time $T_{sim} = 300$; the step size $h = 0.001$. Following [3, 11, 12], the numerical solution of eq. (3.1) involves the following set of equation.

$$X(j) = Y(j-1)h^1 - \sum_{i=1}^{j} c_i^{\,1} X(j-i)$$

$$Y(j) = \left[-\alpha Y(j-1) + F_0 \cos\left(\omega U(j-1)\right) + X(j) - X^3(j) - \beta W(j-1)\right]h^1 - \sum_{i=1}^{j} c_i^{\,1} Y(j-i)$$
(3.2)

$$W(j) = Y(j)h^{q_1} - \sum_{i=1}^{j} c_i^{\,1} W(j-i) \quad U(j) = 1h^1 - \sum_{i=1}^{j} c_i^{\,1} U(j-i)$$

Case (i): In the foregoing system, keeping the forcing amplitude of the sinusoidal excitation term, F_0=0.30 and varying the fractional order q_1, the Duffing system exhibit both chaotic and periodic behaviour, as shown in Figure 1-3. In Figure 1-2, the Duffing system exhibits chaotic dynamics when the fractional order $q_1 = 0.9$ and 0.5 respectively while for q_1=0.1 Duffing system dynamics is regular.



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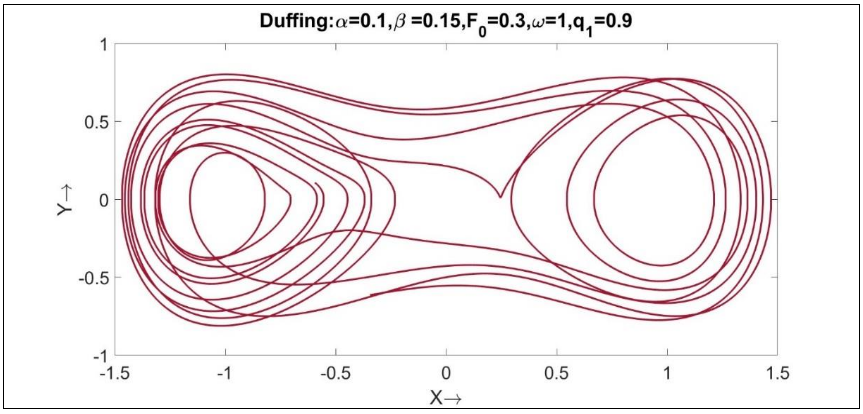
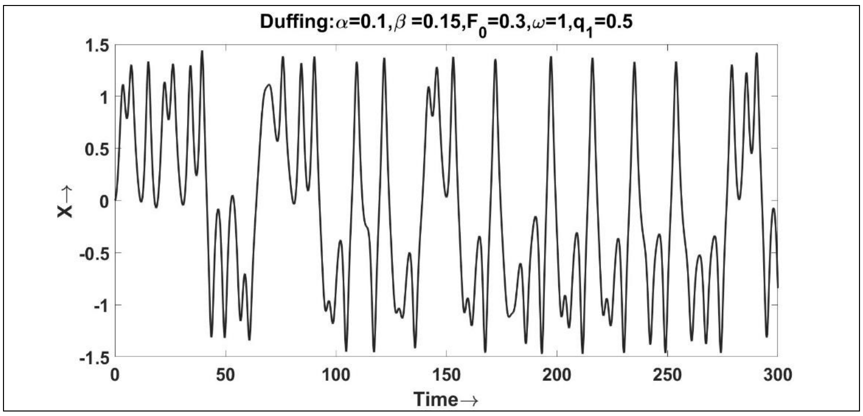


(b)

Fig 1: (a) Time series showing chaotic dynamics of the fractional Duffing system for fractional order α=0.3\alpha = 0.3α=0.3 and (b) Phase portrait indicating chaotic trajectories of the fractional Duffing system for fractional order α=0.3\alpha = 0.3α=0.3



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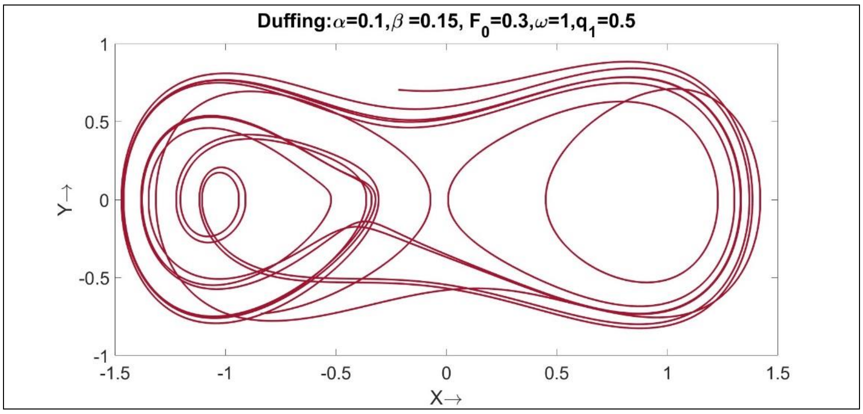
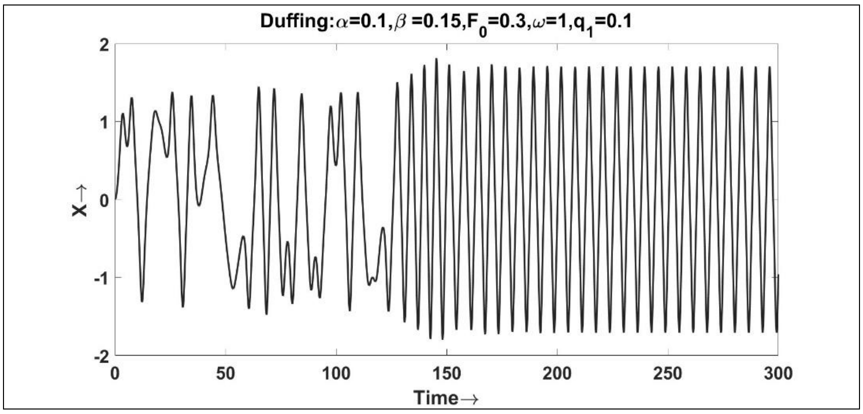


(b)

Fig 2: (a) Time series illustrating chaotic dynamics for fractional order α=0.5\alpha = 0.5α=0.5 and (b) Corresponding phase portrait demonstrating complex chaotic trajectories at fractional order α=0.5\alpha = 0.5α=0.5



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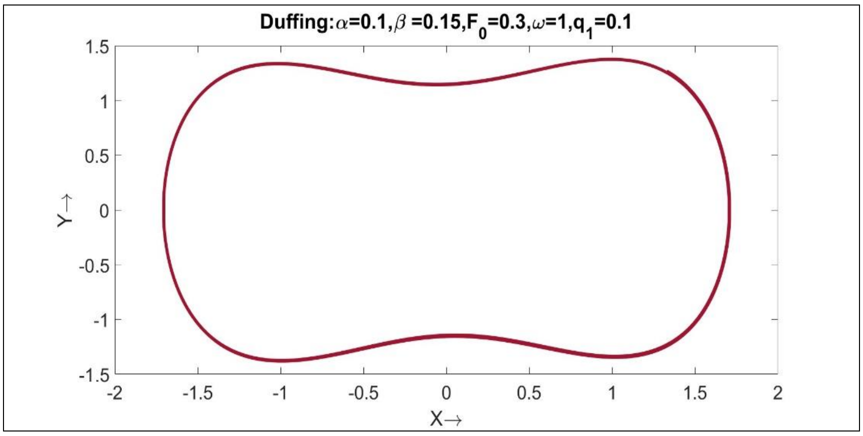


(b)

Fig 3: (a) Time series displaying regular (periodic) dynamics at fractional order α=0.8\alpha = 0.8α=0.8 and (b) Phase portrait clearly showing periodic trajectories in the fractional Duffing system at fractional order α=0.8\alpha = 0.8α=0.8

Table 1: Lyapunov exponents of the fractional duffing system $\alpha = 0.1;\ \beta = 0.15;\ F_0 = F = 0.30;\ \omega = 1.0$

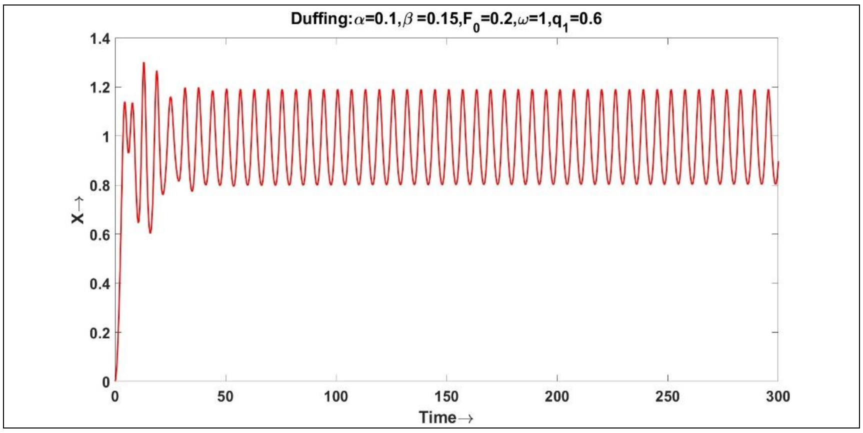
P	L_1	L_2	L_3
0.1	0.0525	-0.0939	-0.2033
0.2	0.1717	-0.3525	-0.5924
0.3	0.1897	-0.5394	-2.0173
0.4	0.1649	-0.5611	-5.4970
0.5	0.2004	-0.5090	-13.7195
0.6	0.0534	-0.3473	-32.5922
0.7	0.1084	-0.3758	-87.2215
0.8	0.1390	-0.3996	-175.9853
0.9	0.1170	-0.3690	-268.8692
0.95	0.1211	-0.3707	-360.8091

An important task in nonlinear dynamics is to quantify also the type of vibration/oscillations i.e., quasi-periodic, random or chaotic. Important feature of chaotic vibrations/oscillations are their sensitivity to initial conditions and among the various methodology adopted to detect chaos is to determine/ estimate the spectrum of Lyapunov exponents (LCE). Geometrically, the maximum of the spectrum of LCE of a dynamical system implies exponential divergence of two very close by trajectories in phase space. We use the algorithm, discussed in detail in [7], to compute the spectrum of LCEs of the fractional dynamical system, eq. (3.1).

For case (i), varying the fractional order of the Duffing system, the spectrum of Lyapunov exponents found using the algorithm [7] are listed in Table 1.

In Table 1, we observe that except for the fractional order value $q_1 = 0.1,\ 0.6$, the maximum of LCE i.e., L_1 are very much greater than 0. Hence the dynamics of the Duffing system for all other remaining cases are term as chaotic as also exhibited in their respective time series and phase portrait. Case (ii)- In this case, as mentioned earlier, we vary the amplitude, F_0, and keep the value of fractional order fixed at $q_1 = 0.6$. Fig.4-6 illustrate the simulated time series and phase portraits computed using eq. (3.2) as regular and chaotic for various parameter values indicated in their respective diagrams.

In case (ii), we find the Duffing system dynamics to be regular in most of the considered instances as indicated by their spectrum of Lyapunov exponents listed in Table:2.



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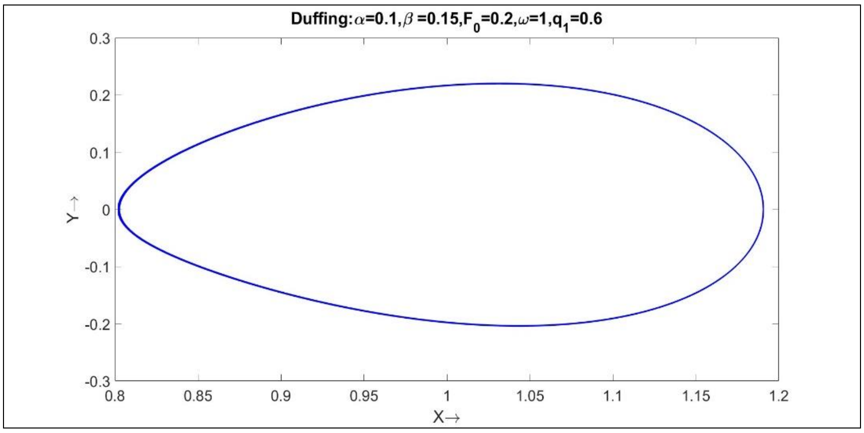
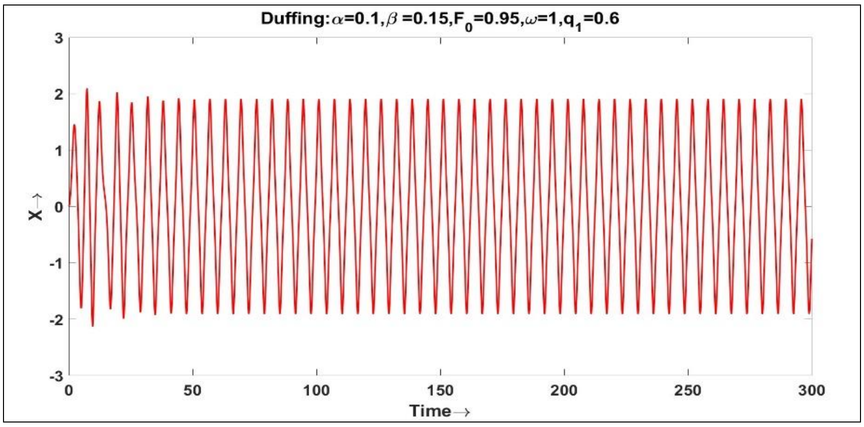


(b)

Fig 4: (a) Time series illustrating regular dynamics of the fractional Duffing system for forcing amplitude f=0.2f = 0.2f=0.2 and (b) Phase portrait depicting periodic trajectories in the fractional Duffing system for forcing amplitude f=0.2f = 0.2f=0.2



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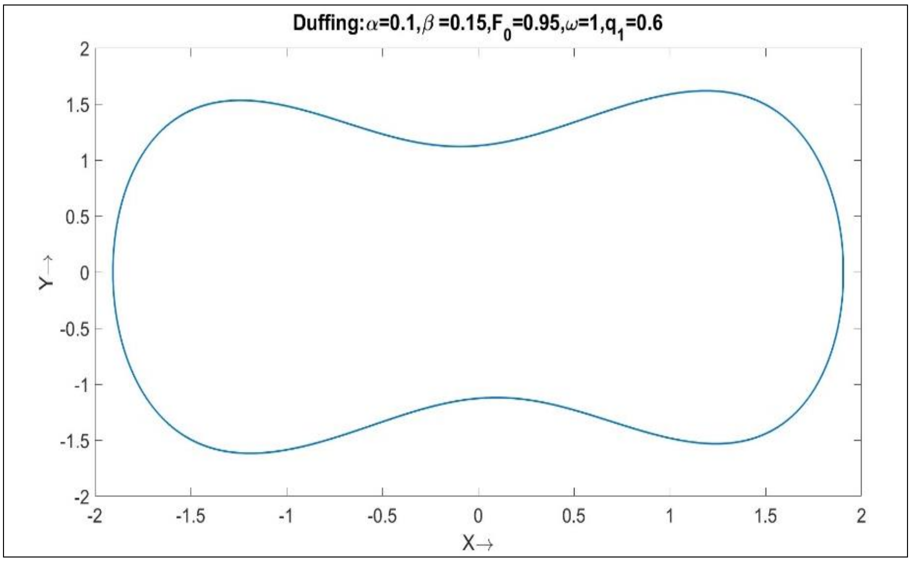
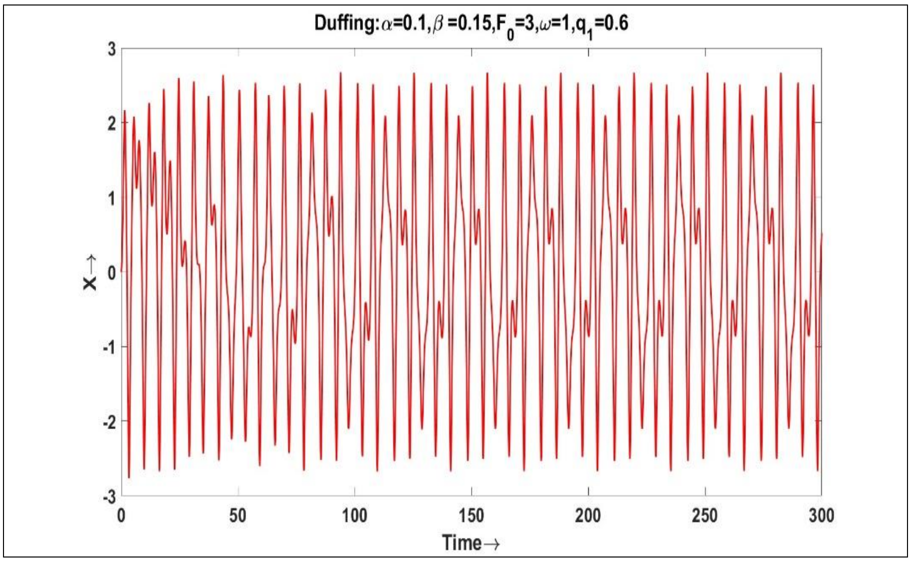


(b)

Fig 5: (a) Time series showing chaotic dynamics of the fractional Duffing system for forcing amplitude f=0.4f = 0.4f=0.4 and (b) Corresponding phase portrait demonstrating chaotic trajectories for forcing amplitude f=0.4f = 0.4f=0.4



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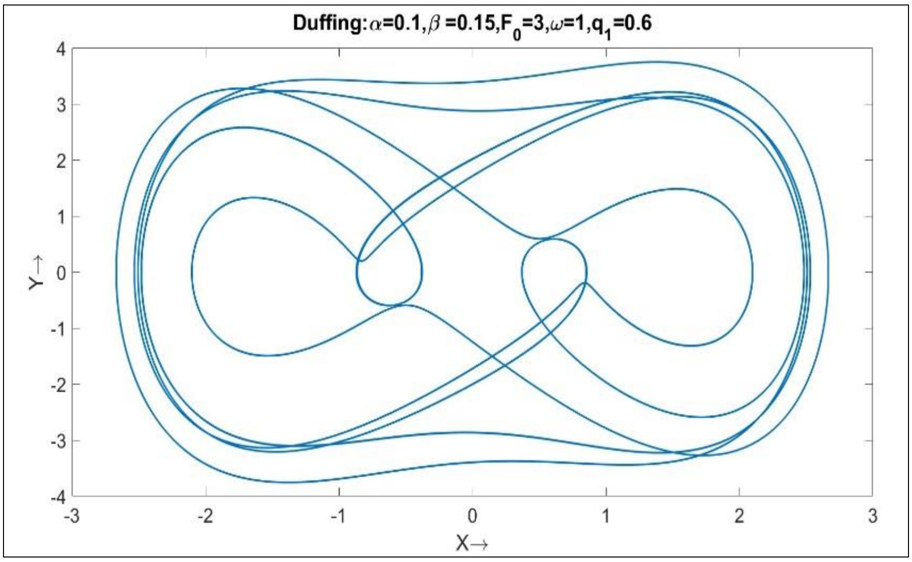

(b)

Fig 6: (a) Time series depicting regular (quasi-periodic) dynamics of the fractional Duffing system for forcing amplitude f=1.5f = 1.5f=1.5 and (b) Phase portrait clearly indicating regular or quasi-periodic trajectories for forcing amplitude f=1.5f = 1.5f=1.5

Table 2: $\alpha = c = 0.1;\ \beta = 0.15;\ p = 0.60;\ \omega = 1.0$

F_o	L_1	L_2	L_3
0.10	-0.0987	-0.1271	-35.3697
0.20	-0.1026	-0.1290	-35.2497
0.30	0.0534	-0.3473	-32.5922
0.40	0.1857	-0.4529	-34.3459
0.50	-0.0957	-0.1363	-35.1107
0.60	-0.0901	-0.1416	-35.0246
0.70	-0.0941	-0.1348	-35.0968
0.80	-0.0937	-0.1334	-35.0878
0.90	-0.0966	-0.1295	-35.0677
0.95	-0.0946	-0.1308	-35.0282
1.00	-0.1005	-0.1250	-35.0015
1.10	-0.0994	-0.1252	-34.9821
1.20	-0.0992	-0.1251	-34.9518
1.30	-0.0796	-0.1465	-34.8162
1.40	-0.0793	-0.1452	-34.8370
1.50	-0.0961	-0.1277	-34.7882
1.80	-0.0973	-0.1263	-34.6012
2.00	-0.0842	-0.1270	-34.7696
2.20	-0.0963	-0.1297	-34.4926
2.40	-0.0371	-0.1913	-34.5575
3.00	+0.0140	-0.2473	-34.1731
3.10	+0.0936	-0.3259	-34.1447
3.20	-0.0827	-0.1228	-34.4256
3.50	-0.0962	-0.1020	-34.3850

4. Results and Discussion

A Duffing system comprising of linear and fractional damping termed is studied for various values of the amplitude of the sinusoidal forcing term. Only two cases are considered for numerical simulation, (1) when the fractional order is varied and (2) when the amplitude of harmonic excitation term is varied. In both the cases, we observe the dynamics to be chaotic and regular or quasi-periodic depending on the considered control parameter of the system. In case (1), we find most of the cases, the maximum of the spectrum of Lyapunov exponents (LCE) are very much greater than zero and hence dynamics corresponds to the chaotic case while in case (2), on varying the amplitude, F_0 and keeping the fractional order $q_1 = 0.6$, the maximum of LCEs are negative and hence for such case the system corresponds to a regular/quasiperiodic behaviour.

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