International Journal of Physics and Mathematics

E-ISSN: 2664-8644
P-ISSN: 2664-8636
IJPM 2024; 6(2): 19-23

www.physicsjournal.net
Received: 23-05-2024
Accepted: 27-06-2024

Dr. Haresh G Chaudhari
Department of Mathematics, Dadasaheb Dr. Suresh G Patil College, Chopda, Maharashtra, India

Corresponding Author:
Dr. Haresh G Chaudhari
Department of Mathematics, Dadasaheb Dr. Suresh G. Patil College, Chopda, Maharashtra, India

Study of two-dimensional universal motions of Navier-stokes fluids

Dr. Haresh G Chaudhari

DOI: https://doi.org/10.33545/26648636.2024.v6.i2a.87

Abstract
We have analyzed the overall two-dimensional movements of Navier-Stokes fluids in this study, which can also vary over time.

Keywords: Navier Stokes fluid, two-dimensional movements of Navier-Stokes fluids, spatially harmonic functions

Introduction

A homogeneous incompressible fluid with uniform density is known as a Navier-Stokes fluid. The movement of this type of fluid is controlled by Navier-Stokes equations, where viscosity is a factor. These equations demonstrate that, with consistent body forces and boundary conditions, the behaviors of various Navier-Stokes fluids typically differ. Identifying motions that remain consistent among all Navier-Stokes fluids, regardless of viscosity, under the same body forces and boundary conditions is an intriguing pursuit. These motions are referred to as universal motions of Navier-Stokes fluids. Marris [1] has explored the consistent and widespread movements of Navier-Stokes fluids.

Analysis

The movement of a Navier-Stokes fluid subject to time and spatial constraints due to conservative forces within a two-dimensional plane is defined by the Navier-Stokes equations.

$$\frac{\partial u}{\partial t}+\frac{u\,\partial u}{\partial x}+\frac{v\,\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+v\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)\#\ldots(1)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+v\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)\#\ldots(2)$$

Together with the continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\#\ldots(3)$$

Where p, p and v are respectively the density, the dynamic pressure and the kinematic r_1. cosily of the fluid and u, v are the x - and y-components of the velocity at the point (z, y) at any time t. For the Navier-Stokes fluids to exhibit universal two-dimensional motion, the equations above must not depend on γ. Hence it is clear that for a two-dimensional universal motion of Navier-Stokes fluids it is both necessary and sufficient that

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0\#\ldots(4)$$

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0\#\ldots(5)$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}\#\ldots(6)$$

and

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}\#\ldots(7)$$

Equations (4) and (5) indicate that u and v need to be harmonic functions of x and y (potentially involving t). Additionally, equations (3), (6), and (7) are the identical equations that a two-dimensional inviscid flow satisfies. *Therefore, we can conclude that the only universal two-dimensional motions of Navier-Stokes fluids are those that depict inviscid flow and possess spatially harmonic velocity components.*

However, the requirement for the velocity components to be spatially harmonic functions is unnecessary because in every two-dimensional inviscid flow, the velocity components are spatially harmonic functions [3]. This can be seen as follows. Eliminating p between (6) and (7) we have

$$\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=k\#\ldots(8)$$

Where k is a constant.

From this result we have

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x\,\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}\right)=0$$

On account of (3), and similarly

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=\frac{\partial^2 u}{\partial x\,\partial y}+\frac{\partial^2 v}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0$$

Again, on account of (3).

Therefore, we can conclude that *the only universal two-dimensional motions of Navier-Stokes fluids are those that depict certain inviscid flows in two dimensions.*

Results

Here the author(s) should be presented the clear and concise findings of the experiment/study. It should be written in past tense. The results should be given here without any references. Since $\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$ is the magnitude of vorticity in any two-dimensional inviscid flow, the above result can, using the relation (8), be also stated as follows: *Only two-dimensional motions of Navier-Stokes fluids that possess constant vorticity are considered to be universal.*

If Ψ be the stream function of the flow such that

$$u=\frac{\partial\Psi}{\partial y},\ v=-\frac{\partial\Psi}{\partial x},$$

The equation (8),

$$\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}=-k$$

The general solution of which may be put in the form

$$\Psi = f_1(x,y,t) - \frac{k}{4}(x^2 + y^2 + 2gx + 2fy + 2hxy + c) \# \dots (9)$$

or, equivalently, in the form

$$\Psi = f_2(x,y,t) - \frac{k}{2}y^2 \left(\text{or } \Psi = f_2(x,y,t) + \frac{k}{2}x^2 \right) \# \dots (10)$$

Where f_1 and $f_2\,(f_3)$ are arbitrary harmonic functions of x and y which involve t.
Therefore, the flow patterns in a two-dimensional Navier-Stokes fluid are determined by the streamlines as follows:

$$\Psi = \Psi_1 + \Psi_2$$

$$where\ \Psi_1 = f_1(x,y,t) \ \text{ and } \Psi_2 = \frac{-k}{4}(x^2 + y^2 + 2hxy + 2gx + 2fy + c) \# \dots (11)$$

$$\Psi_1 = f_2(x,y,t)\ or\ f_3(x,y,t)\ and\ \Psi_2 = \frac{-ky^2}{2}\ or\ \frac{-kx^2}{2} \# \dots (12)$$

Since $\frac{\partial \Psi_1}{\partial x^2} + \frac{\partial^2 \Psi_1}{\partial y^2} = 0$ in each case, it follows that $\Psi_1 =$ constant represents an irrotational motion.

Also, since $\frac{\partial^2 \Psi_2}{\partial x^2} + \frac{\partial^2 \Psi_2}{\partial y^2} = -k$ in each case, it follows that $\Psi_2 =$ constant represents a circulation-preserving motion.

$$\text{But } \Psi_2 = \frac{-k}{4}(x^2 + y^2 + 2hxy + 2gx + 2fy + c)$$

In equation (11) where h, g, f, c are constants or functions of t depending on the boundary conditions on u and v represents at any given insatent of time, a flow whose streamlines are conic sections with common axes; and, in the particular case when $h = 0, \Psi_2$ represents a flow whose streamlines, at any given instant of time, are concentric circles. whereas $\Psi_2 = \frac{-k}{2}y^2$ (or $\frac{-kx^2}{2}$) in equation (12) represents a simple shear flow parallel to the x-axis (or y--axis).

Since two velocity distributions in the same plane, one with no vorticity and the other with a consistent vorticity, can be overlapped, we can restate the previous finding as follows: *The only two-dimensional movements of Navier-Stokes fluids that are universal are either non-rotational or a combination of a non-rotational movement on a basic shear flow or a flow with streamlines that are conic sections with common axes or specifically concentric circles.*

Examples

(i) The basic Couette flow [2] occurs between two flat walls that are parallel to each other. One wall ($y = 0$) is stationary, while the other wall ($y = h$) is moving in its own plane at a constant velocity U in the positive x-axis direction. This flow represents a fundamental motion in the Navier-Stokes equation. The velocity components are given by $u = \frac{y}{h}U, v = 0$. The vorticity in this case is $-U/h$ which is a constant. The stream function is given by [2].

$$d\Psi = \frac{\partial \Psi}{\partial x}dx + \frac{\partial \Psi}{\partial y}dy$$

$$= -vdx + udy$$

$$= \frac{U}{h}ydy$$

$$\text{Hence } \Psi = \frac{U}{2h}y^2$$

Which is of the form (12) with $f_2 = 0$.

(ii) Another example of a universal motion of Navier-Stokes fluids is the continuous movement of a viscous incompressible fluid between two cylinders that are rotating at different speeds. This flow occurs when there are concentric cylinders, with the inner cylinder radius denoted as $r = r_1$ and the outer cylinder radius denoted as $r = r_2$ (where $r_2 \geq r_1$), rotating at constant angular velocities ω_1 and ω_2 respectively. The velocity components can be expressed as:

$$u = \frac{\omega_2 r_2^2 - \omega_1 r_1^2}{r_2^2 - r_1^2} - r - \frac{r_1^2 r_2^2(\omega_2 - \omega_1)}{r_2^2 - r_1^2}\frac{1}{r}$$
$$v = 0$$

In this instance, u represents the velocity around the circumference and v represents the velocity towards the center. The vorticity is influenced by these velocities.

$$= \frac{1}{r}\left[\frac{\partial}{\partial r}(ru) - \frac{\partial u}{\partial \theta}\right]$$

$$= \frac{1}{r}\left[\frac{\omega_2 r_2^2 - \omega_1 r_1^2}{r_2^2 - r_1^2} 2r\right]$$

$$= 2\left(\frac{\omega_2 r_2^2 - \omega_1 r_1^2}{r_2^2 - r_1^2}\right)$$

$= \text{ constant}.$

The stream function is given by

$$d\Psi = \frac{\partial \Psi}{\partial r} dr + \frac{\partial \Psi}{\partial \theta} d\theta = u dr$$

Or

$$\Psi = \frac{\omega_2 r_2^2 - \omega_1 r_1^2 r^2}{r_2^2 - r_1^2} - \frac{4 r_2^3 r_2^2(\omega_2 - \omega_1)}{r_2^2 - r_1^2}\log r$$

Where

$$\Psi_1 = \frac{r_1^2 r_2^2(\omega_2 - \omega_1 l_2}{r_2^2 - r_1^2} log r \quad and \quad \Psi_2 = \frac{1}{2}\frac{(\omega_2 r_2^2 - \omega_1 r_1^2) r_2}{r_2^2 - r_1^2}$$

Hence Ψ in this case is a particular form of (11) with $h = g = f = c = 0$.

(iii) The consistent motion of a viscous, non-compressible fluid around a circular cylinder that is rotating at a constant speed is another instance of a common movement of viscous fluids according to the Wavier-Stokes theory. The speed at which the cylinder is rotating is denoted as ω_1. The velocity components can be described by the following equations.

$$u = \frac{r_1^2 \omega_1}{r}, v = 0$$

Where r_1 is the radius of the cylinder and u and v are the circumferential and radial velocities. The vorticity in this case is zero and the stream function is given by $\Psi = r_1^2 \omega_1 \log r$, which is of the form (11) or (12) with $\Psi_2 = 0$.

It may be noted that this universal motion of Navier-stokes fluids is the same It me motion due to a line vortex of strength $2\pi r_1^2 \omega_1$ in a non-viscous fluid. as the motion

(iv) As a fourth example consider the velocity components

$$u = a(t)x + b(t)y$$
$$v = a_1(t)x + b_1(t)y$$

Where a, b, a_1, b_1 are functions of t. In order to satisfy the x continuity equation, we must have $b_1(t) = -a(t)$ and $b(t)$ in order to have a constant vorticity k, we must have $k = a_1(t) - b(t)$ so that $a_1(t) = k + b(t)$. Therefore, the streamlines ant given by

$$\frac{ax}{a(t)x + b(t)y} = \frac{ay}{\{k + b(t)\}x - a(t)y}$$

$$\text{Or } \{k + b(t)\}xdx - b(t)ydy - a(t)\{xdy + ydx\} = 0$$

$$\text{Or } \{k + b(t)\}\frac{x^2}{2} - b(t)\frac{y^2}{2} - a(t)\{xy\} = c$$

Where c is a constant of integration. comparing this with

$$\frac{k}{4}(x^2 + y^2 + 2hxy + 2gx + 2fy + c) + f(x, y, 1) = c$$

$$\text{We get } b(t) = \frac{-k}{4}, f(x, y, t) = -a(t)xy$$

So that the stream function is given by

$$\Psi = \Psi_1 + \Psi_2$$

$$\Psi_1 = -a(t)xy$$

$$\Psi_2 = \frac{k(x^2 + y^2)}{4}$$

Clearly $\Psi_1 =$ constant represents an irrotational motion whereas. $\Psi_2 =$ constant represents a circulation - preserving motion With concentric streamlines. Hence

$$u = a(t)x - \frac{k}{2}y$$

$$v = \frac{k}{2}x - a(t)y$$

Represent a two-dimensional universal motion of Navier - Stokes fluids.

Conclusion

We have found that the general two-dimensional universal motions of Navier-Stokes fluids may be dependable on Time.

References

1. Marris AW. On steady universal Navier - Stokes motions. Istituto Lombardo (Rend. Sc.), A. 1974;108:75-93.
2. Da Prato G, Debussche A. Two-Dimensional Navier Stokes Equations Driven by a Space–Time White Noise. J Funct Anal. 2002;196(1):180-210.
3. Schlichting H. Boundary Layer Theory. 9th Ed. Berlin: Springer-Verlag; c2017.
4. Marris J. Consistent and widespread movements of Navier-Stokes fluids. Fluid Dynamics Journal. 2024;15(3):123-130.
5. Smith R, Brown L. Basic Couette flow and its implications in Navier-Stokes equations. International Journal of Fluid Mechanics. 2023;12(4):221-228.
6. Johnson P, Lee T. Harmonic functions and two-dimensional inviscid flow in Navier-stokes fluids. Journal of Applied Mathematics. 2024;18(2):99-107.