International Journal of Physics and Mathematics

E-ISSN: 2664-8644
P-ISSN: 2664-8636
IJPM 2024; 6(2): 01-08

www.physicsjournal.net
Received: 01-05-2024
Accepted: 05-06-2024

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On the use of RAHMOH integral transform for solving differential equations

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DOI: https://doi.org/10.33545/26648636.2024.v6.i2a.83

Abstract
In this paper, we introduced a new Laplace-type integral transform called RAHMOH transform which is generalized of Laplace and Sumudu transforms for solving ordinary and partial differential equations. We presented its existence, inverse transform and some essential properties with some theorems and applications.

Keywords: RAHMOH transform, Integral transform, Sumudu transform, laplace transform, differential equations

1. Introduction

The origin of the integral transforms can be traced back to the work of P. S. Laplace in 1780 and Joseph Fourier in 1822. In recent years, there are many transforms obtained from Fourier and Laplace transforms. One of these is RAHMOH transform.

RAHMOH transform is derived from the classical Fourier, Laplace, Sumudu and Natural integral Transforms. In order to solve the differential equations, the integral transform was extensively used.

We will describe the definitions for some of these transforms. Fourier integral transform was defined as:

$$F[f(t)] = f(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-iwt}\, dt \tag{1}$$

The Fourier transform have many applications in physics and engineering processes. The Laplace integral transform is similar with the Fourier transform and is defined as:

$$\mathcal{L}[f(t)] = \int_0^{\infty} f(t) e^{-st}\, dt \tag{2}$$

The Laplace transform is highly efficient for solving some class of ordinary and partial differential equations.

In 1993, Watugala obtained new integral transform called Sumudu integral transform and it is applied in different sciences like physics and engineering recently, also many new transforms based on Sumudu integral transform because of its properties. The mathematical definition of Sumudu transform is

$$S[f(t)](u) = G(u) = \frac{1}{u} \int_0^{\infty} f(t) e^{\frac{-t}{u}}\, dt \tag{3}$$

In 2008, the Natural transform has been introduced. Recently it is applied to many applications in physics and engineers. It is defined as

$$N[f(t)](s,u) = G(s,u) = \frac{1}{u} \int_0^{\infty} f(t) e^{\frac{-st}{u}}\, dt \tag{4}$$

Provided the integral exists for some variables u and s.

In 2019, a new transform has been introduced and it is called Shehu transform, it is derived from Fourier, Laplace, Sumudu and Natural transform. Shehu transform applied to solve ODEs and PDEs and defined as

$$\mathbb{S}[v(t)](r,m) = V(r,m) = \int_0^\infty v(t)e^{\frac{-rt}{m}}\,dt \tag{5}$$

Historically, there are many other transforms also obtained from Fourier, Laplace and Sumudu integral transform. However, most of the existing integral transforms have some limitations and cannot be used directly to solve nonlinear problems or many complex mathematical models.

2. Main Results

2.1 Definition

A new transform called the RAHMOH Transform defined for the function of exponential order; we consider functions in the set A defined by:

$$A = \left\{f(t): \exists M, k_1, k_2 > 0, |f(t)| < Me^{\frac{|t|}{q_i}}, if\ t\epsilon(-1)^i \times [0,\infty)\right\}$$

Which M is constant must be finite number, k_1, k_2 may be finite or infinite.

The RAHMOH transform denoted by the operator $\mathcal{M}(.)$ for the function A by the following integral:

$$\mathcal{M}[f(t)](s,u) = F(s,u) = u^2\int_0^\infty f(t)e^{\frac{-st}{u}}\,dt\ ;\ s > 0, u > 0 \tag{6}$$

It converges if the limit of the integral exists, else, it diverges.

The inverse of RAHMOH transform is given by:

$$\mathcal{M}^{-1}[F(s,u)] = f(t)\ , for\ t \geq 0 \tag{7}$$

where $s\ and\ u$ are RAHMOH transform variables.

Properties of RAHMOH transform

Property 1

Let $\mathcal{M}[f(t)]$ be RAHMOH transform then the following is true:

$$\mathcal{M}[1] = \frac{u^3}{s} \tag{8}$$

$$\mathcal{M}[t] = \frac{u^4}{s^2} \tag{9}$$

$$\mathcal{M}[t^n] = \frac{n!\,u^{n+3}}{s^{n+1}} \tag{10}$$

Proof

Apply in eq (2.1) we get:

(i) $\mathcal{M}[1] = u^2\int_0^\infty e^{\frac{-st}{u}}\,dt = \lim_{k\to\infty}\left[u^2\int_0^k e^{\frac{-st}{u}}\,dt\right] = \lim_{\kappa\to\infty} u^2\left[\frac{-u}{s}e^{\frac{-st}{u}}\right]_0^k = \frac{u^3}{s}$

(ii) $\mathcal{M}[t] = u^2\int_0^\infty t\,e^{\frac{-st}{u}}\,dt = \lim_{k\to\infty}\left[u^2\int_0^k t\,e^{\frac{-st}{u}}\,dt\right]$, integrating by parts to find that:

$$\lim_{k\to\infty} u^2 \left[\frac{u}{s}\int_0^k e^{\frac{-st}{u}}\,dt\right] = \frac{u^4}{s^2}$$

(iii) $\mathcal{M}[t^n] = u^2 \int_0^\infty t^n\, e^{\frac{-st}{u}}\, dt = \lim_{k\to\infty} u^2 \int_0^k t^n\, e^{\frac{-st}{u}}\, dt$, integrating by parts n times and take the limit we find: $\lim_{k\to\infty} u^2 \int_0^k t^n\, e^{\frac{-st}{u}}\, dt = \frac{n!\, u^{n+3}}{s^{n+1}}$

Property 2

Let $\mathcal{M}[f(t)]$ be RAHMOH transform, then the following is true:

(i) $\mathcal{M}[e^{at}] = \frac{u^3}{s-au}$ (11)

(ii) $\mathcal{M}[\sin(at)] = \frac{au^4}{a^2u^2+s^2}$ (12)

(iii) $\mathcal{M}[\cos(at)] = \frac{su^3}{a^2u^2+s^2}$ (13)

(iv) $\mathcal{M}[\sinh(at)] = \frac{au^4}{s^2-a^2u^2}$ (14)

(v) $\mathcal{M}[\cosh(at)] = \frac{su^3}{s^2-a^2u^2}$ (15)

Proof

Applying eq (2.1) we get:

(i) $\mathcal{M}[e^{at}] = \lim_{k\to\infty}\left[u^2 \int_0^k e^{at}\, e^{\frac{-st}{u}}\, dt\right] = \lim_{k\to\infty}\left[u^2 \int_0^k e^{\frac{(ua-s)t}{u}}\, dt\right]$, integrating directly as same as (2.3) in property 1 and take the limit yields:

$$\lim_{k\to\infty}\left[u^2 \int_0^k e^{\frac{(ua-s)t}{u}}\, dt\right] = \frac{u^3}{s-au}$$

(ii) $\mathcal{M}[\sin(at)] = \lim_{k\to\infty}\left[u^2 \int_0^k \sin(at)\, e^{\frac{-st}{u}}\, dt\right]$

Using the fact that:

$$\int_0^\infty e^{mt}\sin(nt)\,dt = \left[\frac{e^{mt}}{m^2+n^2}[\mathrm{m}\sin(nt) - n\cos(nt)]\right]_0^\infty$$

Here taking $m = {-s}/{u}$, $n = a$ and apply it in eq (2.1), so

$$\mathcal{M}[\sin(at)] = \lim_{k\to\infty}\left[u^2\int_0^k \sin(at)\, e^{\frac{-st}{u}}\,dt\right] = \lim_{k\to\infty} u^2 \left[\frac{e^{\frac{-s}{u}t}}{\left(\frac{-s}{u}\right)^2 + a^2}\left[\frac{-s}{u}\sin(at) - a\cos(at)\right]\right]_0^k$$

Apply and take the limit yields:

$$\mathcal{M}[\sin(at)] = \frac{au^4}{a^2u^2 + s^2}$$

$$\mathcal{M}[\cos(at)] = \lim_{k\to\infty}\left[u^2\int_0^k \cos(at)\, e^{\frac{-st}{u}}\, dt\right]$$

Using the fact that:

$$\int_0^\infty e^{mt}\cos(nt)\,dt = \left[\frac{e^{mt}}{m^2+n^2}[\mathrm{n}\sin(nt) + m\cos(nt)]\right]_0^\infty$$

Also taking $m = {}^{-s}/_{u}$, $n = a$ and apply it in eq (2.1), so

$$\mathcal{M}[\cos(at)] = \lim_{k\to\infty}\left[u^2\int_0^k \cos(at)\, e^{\frac{-st}{u}}\, dt\right] = \lim_{k\to\infty} u^2\left[\frac{e^{\frac{-s}{u}t}}{\left(\frac{-s}{u}\right)^2 + a^2}\left[\frac{-s}{u}\sin(at) + a\cos(at)\right]\right]_0^k$$

Apply and take the limit yields:

$$\mathcal{M}[\cos(at)] = \frac{-au^4}{a^2u^2 + s^2}$$

2.2 Theorem: (Existence of RAHMOH)

Let the function $f(t)$ belong to the set A in definition 2.1. *Then its RAHMOH transform $F(s,u)$ exists.*

Proof

Since

$$f(t)\epsilon A = \left\{f(t)\colon \exists M, k_1, k_2 > 0, |f(t)| < Me^{\frac{|t|}{q_i}}, if\ t\epsilon(-1)^i \times [0,\infty)\right\}$$

We take

$$|\mathcal{M}[f(t)]| = F(s,u) = \left|u^2\int_0^\infty f(t)e^{\frac{-st}{u}}\, dt\right|$$

$$\le u^2\int_0^\infty |f(t)|e^{\frac{-st}{u}}\, dt < u^2\int_0^\infty Me^{\frac{|t|}{q_i}}e^{\frac{-st}{u}}\, dt = Mu^2\int_0^\infty e^{\frac{|t|}{q_i}-\frac{st}{u}}\, dt$$

by taking $\frac{1}{q_i} = \gamma$, thus

$$Mu^2\int_0^\infty e^{\frac{|t|}{q_i}-\frac{st}{u}}\, dt = Mu^2\int_0^\infty e^{\frac{(\gamma u - s)t}{u}}\, dt = \frac{Mu^3}{s-\gamma u}$$

Then the RAHMOH exists.

2.3 Theorem: (Linearity property of RAHMOH transform)

Let the functions $\alpha f(t)$ and $\beta g(t)$ be in the set A, then $(\alpha f(t) + \beta g(t)) \in A$, where α and β are nonzero arbitrary constants, and

$$\mathcal{M}[\alpha f(t)+\ \beta g(t)]=\ \alpha\mathcal{M}[f(t)]+\ \beta\mathcal{M}[g(t)]$$

Proof:
Using the Definition 2.1 of RAHMOH transform, we get

$$\mathcal{M}[\alpha f(t)+\ \beta g(t)]=u^2\int_0^\infty[\alpha f(t)+\beta g(t)]\,e^{\frac{-st}{u}}\,dt=u^2\int_0^\infty\left[\alpha f(t)e^{\frac{-st}{u}}+\beta g(t)e^{\frac{-st}{u}}\right]dt=u^2\int_0^\infty\alpha f(t)e^{\frac{-st}{u}}dt+ u^2\int_0^\infty\beta g(t)e^{\frac{-st}{u}}dt=\alpha\left[u^2\int_0^\infty f(t)e^{\frac{-st}{u}}dt\right]+\beta\left[u^2\int_0^\infty g(t)e^{\frac{-st}{u}}dt\right]=\alpha\mathcal{M}[f(t)]\ +\ \beta\mathcal{M}[g(t)]$$

2.4 Theorem: (Change of scale property of RAHMOH transform)
Let the function $f(\beta t)$ be in set A, where β is an arbitrary constant. Then

$$\mathcal{M}[f(\beta t)]=\ \frac{u^2}{\beta}f\left(\frac{s}{\beta},u\right)$$

Proof
Using definition 2.1 of RAHMOH transform, we deduce

$$\mathcal{M}[f(\beta t)]=u^2\int_0^\infty f(\beta t)e^{\frac{-st}{u}}\,dt$$

Substituting $p=\beta t\ \Rightarrow t=\frac{p}{\beta}\ ,dt=\frac{dp}{\beta}\ in\ eq\ (2.13)$ yields

$$\mathcal{M}[f(\beta t)]=\frac{u^2}{\beta}\int_0^\infty f(p)e^{\frac{-sp}{u\beta}}\,dp=\frac{u^3}{\beta}\int_0^\infty f(up)e^{\frac{-sp}{\beta}}\,dp=\ \frac{u^3}{\beta}f\left(\frac{s}{\beta},u\right)$$

2.5 Theorem: (RAHMOH of $f(t)$ Derivatives)
Let $f(t)\in A$ in definition 2.1 and $\mathcal{M}[f(t)]$ its RAHMOH transform, then:

(i) $\mathcal{M}[f'(t)]=\frac{s}{u}F(s,u)-u^2f(0)$

(ii) $\mathcal{M}[f''(t)]=\frac{s^2}{u^2}F(s,u)-sf(0)-u^2f'(0)$

(iii) $\mathcal{M}\left[f^{(k)}(t)\right]=\frac{s^k}{u^k}F(s,u)-u^2\sum_{n=0}^{k-1}\left(\frac{s}{u}\right)^{k-(n+1)}f^{(n)}(0)$

Proof: It is obtained by applying in definition 2.1 and using integration by parts.

The following properties are obtained using Leibniz's rule

$$\mathcal{M}\left[\frac{\partial f(x,t)}{\partial x}\right]=u^2\int_0^\infty\frac{\partial f(x,t)}{\partial x}\,e^{\frac{-st}{u}}\,dt=\frac{\partial}{\partial x}\left[u^2\int_0^\infty f(x,t)e^{\frac{-st}{u}}\,dt\right]$$

$$=\frac{\partial}{\partial x}[F(x,s,u)]=\ \frac{d}{dx}[F(x,s,u)]$$

$$\mathcal{M}\left[\frac{\partial^n f(x,t)}{\partial x^n}\right]=u^2\int_0^\infty\frac{\partial^n f(x,t)}{\partial x^n}\,e^{\frac{-st}{u}}\,dt=\frac{\partial^n}{\partial x^n}\left[u^2\int_0^\infty f(x,t)e^{\frac{-st}{u}}\,dt\right]$$

$$=\frac{\partial^n}{\partial x^n}[F(x,s,u)]=\frac{d^n}{dx^n}[F(x,s,u)]$$

3. Some Applications

RAHMOH transform can be used to solve Linear and non-Linear ODEs and PDEs. The following simple examples show that:

3.1 Example: Consider the following first order ordinary differential equation:

$$\frac{df(t)}{dt} + f(t) = 0,$$

Subject to initial condition

$$f(0) = 1$$

Solution:

By taking RAHMOH transform

$$\mathcal{M}\left[\frac{df(t)}{dt}\right] + \mathcal{M}[f(t)] = 0$$

Using eq (2.14) yields

$$\frac{s}{u}F(s,u) - u^2 f(0) + F(s,u) = 0$$

Using initial condition, then

$$F(s,u) = \frac{u^3}{s+u}$$

By taking RAHMOH inverse of eq (3.3) yields $f(t) = e^{-t}$

3.2 Example

Consider the following second order non-homogeneous ODE:

$$\frac{d^2 f(t)}{dt^2} + \frac{df(t)}{dt}$$

Subject to the initial conditions

$$f(0) = 0\,, \frac{df(0)}{dt} = 0$$

Solution

Applying the RAHMOH transform on both sides of eq (3.4), we get

$$\frac{s^2}{u^2}F(s,u) - sf(0) - u^2 f'(0) + \frac{s}{u}F(s,u) - u^2 f(0) = \frac{2u^3}{s}$$

Substituting the given initial conditions and simplify, we get

$$F(s,u) = -\frac{2u^3}{s} + \frac{2u^4}{s^2} + \frac{2u^3}{s+u}$$

Take RAHMOH transform we obtain: $f(t) = -2 + 2t + 2e^{-t}$

3.3 Example

Consider the following homogeneous PDE:

$$\frac{\partial f(x,t)}{\partial t} = \frac{\partial^2 f(x,t)}{\partial x^2}$$

Subject to the boundary and initial conditions:

$$f(0,t) = 0, f(1,t) = 0, f(x,0) = 3\sin(2\pi x)$$

Solution

Applying the RAHMOH transform on both sides of eq (3.6), we get

$$\frac{s}{u}F(x,s,u) - u^2 f(x,0) = \frac{d^2[F(x,s,u)]}{dx^2}$$

Substituting the given initial conditions and simplify, we get

$$\frac{d^2[F(x,s,u)]}{dx^2} - \frac{s}{u}\,F(x,s,u) = -3u^2\sin(2\pi x)$$

The general solution of eq (3.8) can be written as:

$$F(x,s,u) = F_h(x,s,u) + F_p(x,s,u)$$

Where $F_h(x,s,u)$ is the solution of the homogeneous part which is given by

$$F_h(x,s,u) = \alpha_1 e^{\sqrt{\frac{s}{u}}x} + \alpha_2 e^{-\sqrt{\frac{s}{u}}x}$$

And $F_p(x,s,u)$ is the solution of the nonhomogeneous part which is given by

$$F_p(x,s,u) = \beta_1\sin(2\pi x) + \beta_2\cos(2\pi x)$$

Applying the boundary conditions on eq (3.10) yields

$$\alpha_1 + \alpha_2 = 0\,, and\ \alpha_1 e^{\sqrt{\frac{s}{u}}} + \alpha_2 e^{-\sqrt{\frac{s}{u}}} = 0 \Rightarrow F_h(x,s,u) = 0\,, since\ \alpha_1 = \alpha_2 = 0$$

Using the method of undetermined coefficients on the nonhomogeneous part, we get

$$F_p(x,s,u) = \frac{3u^3}{s+4\pi^2 u}\sin(2\pi x)$$

Then eq (3.9) become

$$F(x,s,u) = \frac{3u^3}{s+4\pi^2 u}\sin(2\pi x)$$

Taking the inverse RAHMOH transform for eq (3.13), we get

$$f(x,t) = 3e^{-4\pi^2 t}\sin(2\pi x)$$

Conclusion

We introduced an efficient Laplace-type integral transform called the RAHMOH transform for solving both ordinary and partial differential equations. We presented its existence, inverse transform and some essential properties. We conclude that RAHMOH is highly efficient because of the following Pros:

It is generalized of the Laplace and Sumudu transforms.

RAHMOH transform become Laplace transform when the variable $u = 1$.
For advanced research in physical science and engineering, the proposed integral transform can be considered a stepping-stone to the Sumudu transform, the natural transform, and the Laplace transform.
The relation between RAHMOH transform and Shehu transform is

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