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# Use of various finite difference methods for solving PDE

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**Abstract**
The finite difference method has long been a standard numerical approach for solving partial differential equations. However, its widespread application is accompanied by inherent limitations affecting accuracy and efficiency. This research will compare the accuracy of various method like Bender-Schmidt Method, Crank -Nicholson Difference Method, Laasonen Method and Du Fort & Frankel Method, in completing numerical solutions of partial differential equations, which is limited to certain boundary condition.

**Keywords:** Finite difference method, explicit method, central difference & implicit method

## Introduction
There are many problems in the field of science, engineering and technology which can be solved by partial differential formulation. The simplest and historically oldest method, known as the finite difference method (FDM) comprises of marking grid points within the domain and the derivatives are approximated by difference methods. For example for a function f(x) the derivative is approximated as $\frac{f(x+h)-f(x)}{h}$, where $h$ is the distance between grid points. In this method we try to obtain the approximate solution such that it is close to the true solution at the grid points. The finite difference methods are ill-suited to deal with complex geometries, both in terms of computational domains and discontinuities in the solution.

## Finite Difference Methods
Given a slender rod of unit length, its temperature can be explained by the 1-D heat equation

$$u_t = \alpha u_{xx} \text{ for } 0 < x < 1, 0 < t < T \quad (1.01)$$

For a rod of length $L$, a simple change of variables can be used to transform the problem to the interval [0,1].
The initial condition temperature is assigned for each point in the rod

$$u(x, 0) = f(x) \text{ for } 0 < x < 1 \quad (1.02)$$

In addition, data must be given explaining changes occurring at each end of the rod. When at a particular temperature the rod is kept (by immersing each end in a fluid bath with a temperature that fluctuates with time, for example), then the boundary conditions

$$u(0,t) = g_0(t), u(1,t) = g_1(t) for\ 0 < t \quad (1.03)$$

Other physical situations give different forms for the boundary conditions. For example, keeping an end of the rod insulated corresponds to specifying that the partial; derivative $u_x$ is zero at that end of the rod: conditions

$$u(0,t) = 0, u(1,t) = 0 \quad (1.04)$$

A third possibility is that an end of the rod is subject to convective cooling. The corresponding boundary condition involves a combination of $u$ and $u_x$ at the appropriate end

We divide the interval $[0,1]$ into $n$ pieces, each of length $h = \Delta x = {}^{1}/_{n}$. The correspondinggrid points are denoted by$x_i$,for $i = 0, \dots, n$. The ends of the interval are at $x_0 = 0 \quad and\ x_n = 1$; the interior points are $x_i = ih,$
for $i = 1, \dots, n-1.$

In similar manner, we define a mesh for the time interval, with $m$ subdivisions

with $k = \Delta t = {}^{T}/_{m}$ and $t_j = jk, j = 0,1, \dots m.$ As with the variable $x,$ the ends of the time interval are$t_0 = 0\ and\ t_m = T.$

The solution at a grid point $u(x_i, t_j)$ is denoted$u_{i,j}$. Similarly, values of the initial condition $f(x)$ at grid points are abbreviated as $f_i$, and values of boundary conditions are either $g_{0j}$ or $g_{1j}$. Finite –difference techniques replace the partial derivatives in the PDE with difference quotients. For heat equation, we use the forward difference formula for $u_t$:

$$u_t = \frac{1}{k}\left[u_{i,j+1} - u_{i,j}\right] \tag{1.05}$$

Similarly, we replace the second derivative by the finite – difference formula using the fact that the spacing between points in the $x$-direction is$h$. If this difference formula is applied at the $j^{th}$ time step, we have

$$\alpha u_{xx} = \frac{c}{h^2}\left[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right] \tag{1.06}$$

This expression yield an explicit method that is easy to solve, but imposes restriction on the relative values of the mesh spacing in the $x$- and $t$- directions (to maintain stability of the solution, $i.e.$, to prevent the inevitable errors in the solution from becoming larger as the calculations proceed from $t = 0 \quad to \quad t = T$).

**Schmidt Method**

The method (1.06) is called the Schmidt method or forward difference approximation for time derivative and central difference approximation for the space derivative. The method gives a relation between the function values at the two levels $j+1$ and j and is called a two level formula. The solution value at any point$(i, j+1)$on the $(j+1)th$ level is expressed in terms of the solution values at the points$(i-1, j)(i, j)$and$(i+1, j)$ on the $i$ th level. Such method is called an explicit method. The truncation error $T$ at the node$(i, j+1)$ is given

$$T_{i,j} = u_{i,j+1} - u_{i,j} - r(u_{i+1,j} - 2u_{i,j} + u_{i-1,j} \tag{1.07}$$

Substituting Taylor series expansion of each term about $(x_i, t_j)$on the right side of (1.07) and simplifying we get

$$T_{i,j} = k\frac{\partial u_{i,j}}{\partial t} + \frac{k^2}{2}\frac{\partial^2 u_{i,j}}{\partial t^2} + \frac{k^3}{6}\frac{\partial^3 u_{i,j}}{\partial t^3} \quad + \cdots - r\left(h^2\frac{\partial^2 u_{i,j}}{\partial x^2} + \frac{h^4}{12}\frac{\partial^4 u_{i,j}}{\partial x^4} + \frac{h^6}{360}\frac{\partial^6 u_{i,j}}{\partial x^2} + \cdots\right)$$

$$= k\left(\frac{\partial u_{i,j}}{\partial t} - \alpha\frac{\partial^2 u_{i,j}}{\partial x^2}\right) + \frac{k^2}{2}\frac{\partial^2 u_{i,j}}{\partial t^2} + \frac{k^3}{6}\frac{\partial^3 u_{i,j}}{\partial t^3} - \frac{k\alpha h^4}{12}\frac{\partial^4 u_{i,j}}{\partial x^4} - \frac{k\alpha h^6}{360}\frac{\partial^6 u_{i,j}}{\partial x^2} \dots \tag{1.08}$$

Using equation(1.01) in equation (1.08), we obtained

$$T_{i,j} = \frac{kh^2c^2}{2}\left(r - \frac{1}{6}\right)\frac{\partial^4 u_{i,j}}{\partial x^4} + \frac{kc^2h^4}{6}\left(r^2 - \frac{1}{60}\right)\frac{\partial^6 u_{i,j}}{\partial x^6} + \cdots \tag{1.09}$$

or

$$\frac{1}{k}T_{i,j} = \frac{h^2c^2}{2}\left(r - \frac{1}{6}\right)\frac{\partial^4 u_{i,j}}{\partial x^4} + \frac{c^2h^4}{6}\left(r^2 - \frac{1}{60}\right)\frac{\partial^6 u_{i,j}}{\partial x^6} + \cdots \tag{1.10}$$

Therefore the method is of the order$k + h^2$. When $r = {}^{1}/_{6}$, the method is of the order $k^2 + h^4$.

**Crank-Nicolson Method**

If we replace $\frac{\partial u}{\partial t}$ by forward difference approximation and $\frac{\partial^2 u}{\partial x^2}$ by average of central difference in space at $j$ and $j+1$ level, after simplifying Eq(1.01) becomes

$$\frac{u_{i,j+1}-u_{i,j}}{\Delta t} = \alpha \frac{1}{2}\left(\frac{u_{i-1,j}-2u_{i,j}-u_{i+1,j}}{(\Delta x)^2} + \frac{u_{i-1,j+1}-2u_{i,j+1}+u_{i+1,j+1}}{(\Delta x)^2}\right) \tag{1.11}$$

It can be written as

$$-\frac{r}{2}u_{i-1,j+1} + (1+r)u_{i,j+1} - \frac{r}{2}u_{i+1,j+1} = \frac{r}{2}u_{i-1,j} + (1-r)u_{i,j} + \frac{r}{2}u_{i+1,j} \tag{1.12}$$

The truncation error

$$T_{i,j} = u_{i,j+1} - u_{i,j} - \frac{r}{2}(u_{i-1,j} - 2u_{i,j} - u_{i+1,j} + u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}) \tag{1.13}$$

$$= k\left(\frac{\partial u_{i,j}}{\partial t} - \alpha\frac{\partial^2 u_{i,j}}{\partial x^2}\right) + \frac{1}{2}k^2\frac{\partial}{\partial t}\left(\frac{\partial u_{i,j}}{\partial t} - \alpha\frac{\partial^2 u_{i,j}}{\partial x^2}\right) - \frac{k\alpha h^2}{12}\frac{\partial^4 u_{i,j}}{\partial x^4} - \frac{k^3\alpha}{4}\frac{\partial^4 u_{i,j}}{\partial x^2 \partial t^2} - \cdots \tag{1.14}$$

Using Eq(1.01), we find

$$k^{-1}T_{i,j} = \left(\frac{\partial u_{i,j}}{\partial t} - \alpha\frac{\partial^2 u_{i,j}}{\partial x^2}\right) + \frac{1}{2}k\frac{\partial}{\partial t}\left(\frac{\partial u_{i,j}}{\partial t} - \alpha\frac{\partial^2 u_{i,j}}{\partial x^2}\right) - \frac{\alpha h^2}{12}\frac{\partial^4 u_{i,j}}{\partial x^4} - \frac{k^2\alpha}{4}\frac{\partial^4 u_{i,j}}{\partial x^2 \partial t^2} - \cdots \tag{1.15}$$

Thus the order of this method is $O(k^2 + h^2)$.

The Crank-Nicolson method is unconditionally stable and has better truncation error, $O(h^2 + k^2)$, than the basic implicit method.

**Laasonen Method**

In Eq(1.01), first-order derivative is replaced by backward difference in time as well as second order is replaced by central difference in space. We get

$$\frac{u_{i,j}-u_{i,j-1}}{\Delta t} = \alpha\frac{u_{i,j-1}-2u_{i,j}-u_{i+1,j}}{(\Delta x)^2} \tag{1.16}$$

After rearranging equation and at time step $(j+1)^{th}$ level equation is given by

$$-ru_{i-1,j+1} + (1+2r)u_{i,j+1} - ru_{i+1,j+1} = u_{i,j} \tag{1.17}$$

where $r = \frac{\propto \Delta t}{(\Delta x)^2} = \frac{\propto k}{h^2}$ is called mesh ratio parameter.

The schematic form (1.17) is called Laasonen Method, which is also called two level scheme since $j\,th$ and $(j+1)th$. The solution value at any point $(i, j+1)$on the $(j+1)th$ level is dependent on the solution values at the neighboring points on the same level and on one value on the $j\,th$ level. Since the solution values at the $(j+1)th$ level are obtained implicitly, the method (1.17) is called an implicit method. The truncation error

$$T_{i,j} = -ru_{i-1,j+1} + (1+2r)u_{i,j+1} - ru_{i+1,j+1} - u_{i,j} \tag{1.18}$$

Expanding each term on the right hand side of Eq(1.18) about $(x_i, t_j)$ and simplifying, we get

$$T_{i,j} = k\left(\frac{\partial u_{i,j}}{\partial t} - \alpha\frac{\partial^2 u_{i,j}}{\partial x^2}\right) + \frac{k^2}{2}\frac{\partial^2 u_{i,j}}{\partial t^2} - k^2\alpha\frac{\partial^3 u_{i,j}}{\partial x^2 \partial t} - \frac{k\alpha h^2}{12}\frac{\partial^4 u_{i,j}}{\partial x^4} - \frac{k^3\alpha}{2}\frac{\partial^4 u_{i,j}}{\partial x^2 \partial t^2} \ldots\ldots\ldots \tag{1.19}$$

If we use the relation Eq(1.01), then Eq(1.19) becomes

$$k^{-1}T_{i,j} = \frac{k}{2}\left(\frac{\partial^2 u_{i,j}}{\partial t^2} - 2\alpha\frac{\partial^3 u_{i,j}}{\partial x^2 \partial t}\right) - \frac{\alpha h^2}{12}\frac{\partial^4 u_{i,j}}{\partial x^4} - \cdots \quad (1.20)$$

Thus the order of this method is $O(k + h^2)$.

**DuFort-Frankel Method**

The explicit Richardson method three level is

$$u_{i,j+1} = u_{i,j-1} + 2r\left(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right) \quad (1.21)$$

If we replace $u_{i,j}$ in Richardson method by the mean of the values $u_{i,j+1}$ and $u_{i,j-1}$

$$u_{i,j} \approx \frac{1}{2}\left(u_{i,j+1} + u_{i,j-1}\right) \quad (1.22)$$

Then we get

$$u_{i,j+1} = u_{i,j-1} + 2r\left(u_{i-1,j} - u_{i,j+1} - u_{i,j-1} + u_{i+1,j}\right) \quad (1.23)$$

which on simplification can be written as

$$u_{i,j+1} = \frac{(1-2r)}{1+2r}u_{i,j-1} + \frac{2r}{1+2r}\left(u_{i-1,j} + u_{i+1,j}\right) \quad (1.24)$$

This difference scheme is called the DuFort and Frankel method. It is an explicit three level method. The truncation error of the method is given by

$$T_{i,j} = (1 + 2r)u_{i,j+1} - (1 - 2r)u_{i,j-1} - 2r\left(u_{i-1,j} + u_{i+1,j}\right) \quad (1.25)$$

Expanding in Taylor's series the terms about the point $(x_i, t_j)$, we have

$$T_{i,j} = (1 + 2r)\left(u_{i,j} + k\frac{\partial u_{i,j}}{\partial t} + \frac{k^2}{2}\frac{\partial^2 u_{i,j}}{\partial t^2} + \cdots\right)$$

$$-(1 - 2r)\left(u_{i,j} - k\frac{\partial u_{i,j}}{\partial t} + \frac{k^2}{2}\frac{\partial^2 u_{i,j}}{\partial t^2} + \cdots\right)$$

$$-2r\left(2u_{i,j} + h^2\frac{\partial^2 u_{i,j}}{\partial x^2} + \frac{h^4}{12}\frac{\partial^4 u_{i,j}}{\partial x^4} + \cdots\right)$$

$$= 2k\frac{\partial u_{i,j}}{\partial t} + 2rk^2\frac{\partial^2 u_{i,j}}{\partial t^2} - 2rh^2\frac{\partial^2 u_{i,j}}{\partial x^2} - \frac{rh^4}{6}\frac{\partial^4 u_{i,j}}{\partial x^4} + \cdots \quad (1.26)$$

and

$$k^{-1}T_{i,j} = 2\left(\frac{\partial u_{i,j}}{\partial t} - \alpha\frac{\partial^2 u_{i,j}}{\partial x^2}\right) + 2\left(\frac{k}{h}\right)^2\alpha\frac{\partial^2 u_{i,j}}{\partial t^2} - \alpha\frac{h^2}{6}\frac{\partial^4 u_{i,j}}{\partial x^4} + \frac{k^2}{3}\frac{\partial^3 u_{i,j}}{\partial t^3} + \cdots \quad (1.27)$$

Thus we have

$$k^{-1}T_{i,j} \to 0 \; if \; \frac{k}{h} \to 0 \; as \; h \to 0$$

and

$$\frac{\partial u_{i,j}}{\partial t} = \alpha\frac{\partial^2 u_{i,j}}{\partial x^2}$$

Thus the difference scheme is consistent with the difference equation $u_t = \alpha u_{xx}$ only when $\frac{k}{h} \to 0 \; as \; h \to 0$. In this case, the order of method is $k^2 + h^2 + \left(\frac{k}{h}\right)^2$.

**Numerical Problem:**
Solve the heat conduction equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \tag{1.28}$$

subject to the initial and boundary conditions

$$u(x,0) = sin\pi x \quad 0 \le x \le 1 \tag{1.29}$$

$$u(0,t) = u(1,t) = 0 \tag{1.30}$$

Using the following methods
i) Schmidt Method
ii) Crank-Nicolson Method
iii) Laasonen Method
iv) DuFort-Frankel Method

For $h = {}^1/_5$ and $k = 0.01$, calculateupto two time levels and compare the results

with the theoretical solutions$u(x,t) = e^{-\pi^2 t} sin\pi x$.

**Solution:** The nodal points are given by

$$x_i = ih, i = 0,1,2,3,4,5 \quad t_j = jk,$$

$$j = 0,1,2, \ldots$$

The boundary conditions are given by

$u_{0,j} = 0 = u_{5,j}$ , $j = 0,12, \ldots$

**Schmidt Method**

$$u_{0,0} = 0, \quad u_{1,0} = 0.5877,$$

$$u_{2,0} = 0.9510, \quad u_{3,0} = 0.9510,$$

$$u_{4,0} = 0.5877, \quad u_{5,0} = 0$$

For,$= {}^1/_4$, the Schmidt method simplifies to $u_{i,j+1} = \frac{1}{4}\left(u_{i-1,j} + 2u_{ij} + u_{i+1,j}\right)$

We obtain
For $j = 0; i = 1,2,3,4$

$$u_{1,1} = 0.5316, u_{2,1} = 0.8601,$$

$$u_{3,1} = 0.8601, u_{4,1} = 0.5316$$

For $j = 1; i = 1,2,3,4$

$$u_{1,2} = 0.48105, u_{2,2} == 0.7779, u_{3,2} = 0.7779, u_{4,2} = 0.48105$$

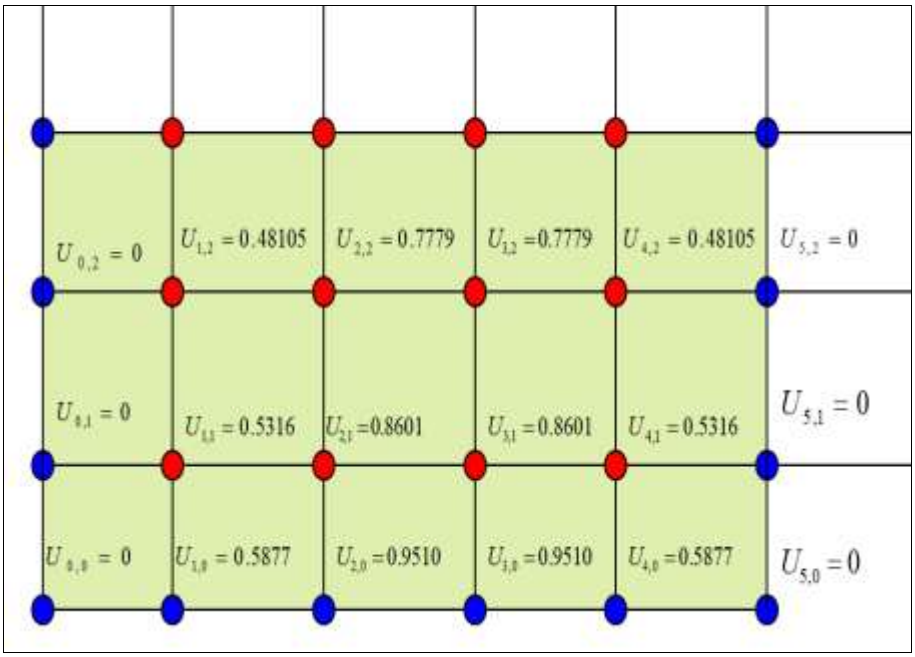


**Fig 1:** Representation of nodal values of Schmidt Method

**Table 1:** Schmidt Method Results at Nodal Points

| x | Analytical Solutions | Schmidt Method Results | Absolute Error |
|---|---|---|---|
| 0.2 | 0.4824 | 0.4810 | 0.0014 |
| 0.4 | 0.7806 | 0.7779 | 0.0027 |
| 0.6 | 0.7806 | 0.7779 | 0.0027 |
| 0.8 | 0.4824 | 0.4810 | 0.0014 |
| Sum of Absolute Error | | | 0.0082 |

**Crank-Nicolson Method**

$$-\frac{1}{8}u_{i-1,j+1} + \frac{5}{4}u_{i,j+1} - \frac{1}{8}u_{i+1,j+1} = \frac{1}{8}u_{i-1,j} + \frac{3}{4}u_{i,j} + \frac{1}{8}u_{i+1,j}$$

We obtain

For $j = 0; i = 1,2,3,4$

$u_{1,1} = 0.5341,\ u_{2,1} = 0.8642,\ u_{3,1} = 0.8642,\ u_{4,1} = 0.5341$

$u_{1,2} = 0.4854,\ u_{2,2} = 0.7854,\ u_{3,2} = 0.7854,\ u_{4,2} = 0.4854$

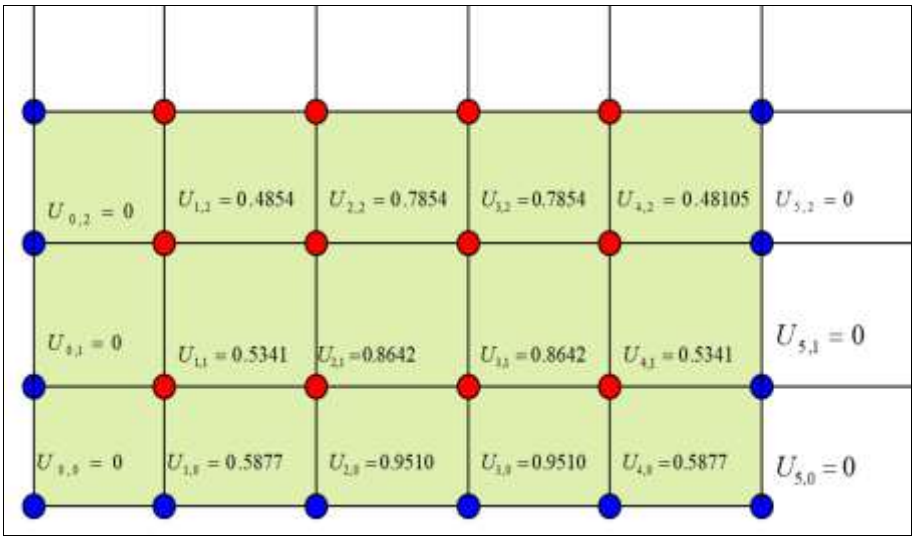


**Fig 2:** Representation of nodal values of Crank-Nicolson Method

**Table 2:** Crank-Nicolson Method Results at Nodal Points

| x | Analytical Solutions | Crank-Nicolson Method Results | Absolute Error |
|---|---|---|---|
| 0.2 | 0.4824 | 0.4854 | 0.0030 |
| 0.4 | 0.7806 | 0.7854 | 0.0048 |
| 0.6 | 0.7806 | 0.7854 | 0.0048 |
| 0.8 | 0.4824 | 0.4854 | 0.0030 |
| Sum of Absolute Error | | | 0.0156 |

**Laasonen Method**

The Laasonen Method simplifies to

$$-\frac{1}{4}u_{i-1,j+1}+\frac{3}{2}u_{i,j+1}-\frac{1}{4}u_{i+1,j+1}=u_{i,j}$$

We obtain

For $j=0; i=1,2,3,4$

$$u_{1,1}=0.5365,\ u_{2,1}=0.8681,\ u_{3,1}=0.8681,\ u_{4,1}=0.5365$$

For j=1; $i=1,2,3,4$

$$u_{1,2}=0.4897,\ u_{2,2}=0.7924,\ u_{3,2}=0.7924,\ u_{4,2}=0.4897$$

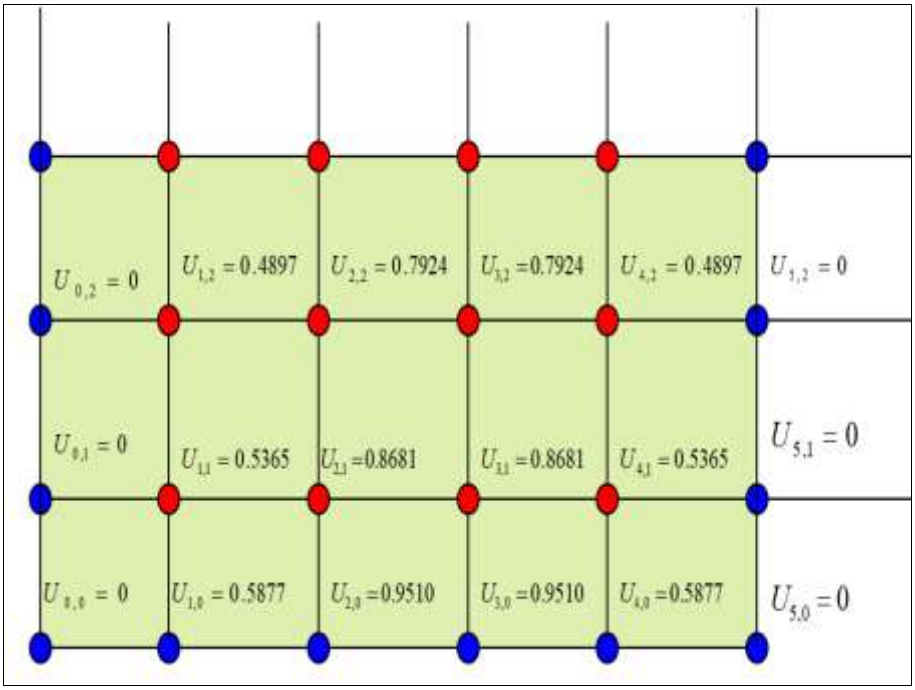


**Fig 3:** Representation of nodal values of Laasonen Method

**Table 3:** Laasonen Method Results at Nodal Points

| x | Analytical Solutions | Laasonen Method Results | Absolute Error |
|---|---|---|---|
| 0.2 | 0.4824 | 0.4897 | 0.0073 |
| 0.4 | 0.7806 | 0.7924 | 0.0118 |
| 0.6 | 0.7806 | 0.7924 | 0.0118 |
| 0.8 | 0.4824 | 0.4897 | 0.0073 |
| Sum of Absolute Error | | | 0.0382 |

**DuFort-Frankel Method**

The DuFort-Frankel Method simplifies to

$$u_{i,j+1}=\frac{1}{3}\left[u_{i,j-1}+u_{i+1,j}+u_{i-1,j}\right]$$

We need the solution on the first level to start the computation. We use the results on the first level obtained by the Schmidt method.

$$u_{1,2} = 0.4826,\ u_{2,2} = 0.7809,\ u_{3,2} = 0.7809,\ u_{4,2} = 0.4826$$

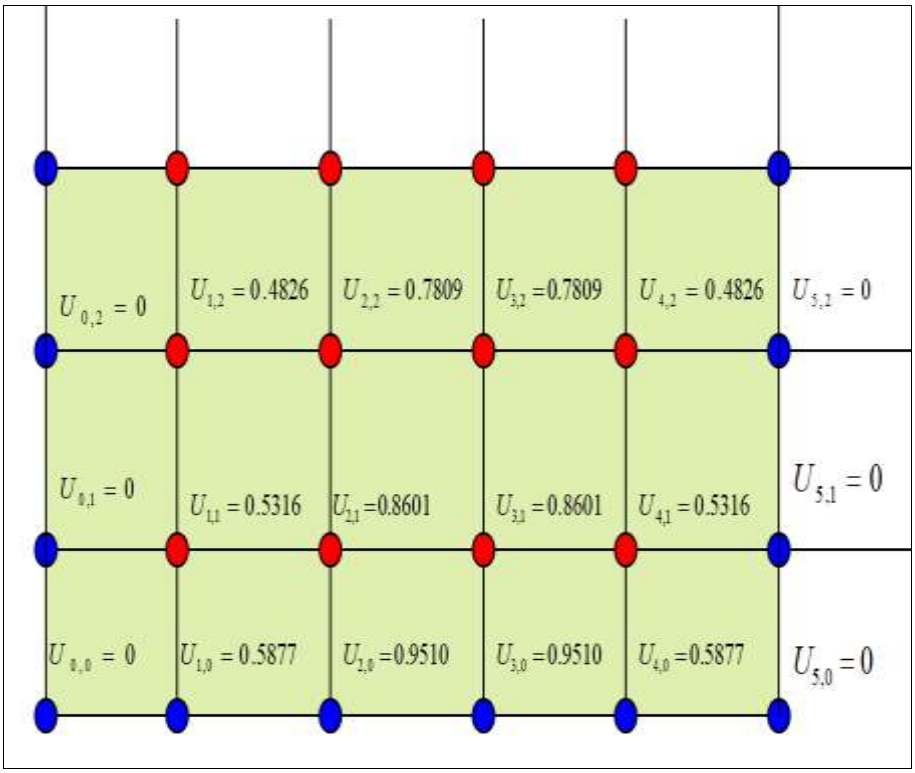


**Fig 4:** Representation of nodal values of DuFort-Frankel Method

**Table 4:** DuFort-Frankel Method Results at Nodal Points

| x | Analytical Solutions | DuFort-Frankel Method Results | Absolute Error |
|---|---|---|---|
| 0.2 | 0.4824 | 0.4826 | 0.0002 |
| 0.4 | 0.7806 | 0.7809 | 0.0003 |
| 0.6 | 0.7806 | 0.7809 | 0.0003 |
| 0.8 | 0.4824 | 0.4826 | 0.0002 |
| Sum of Absolute Error | | | 0.001 |

## Conclusion

From Table 1, Table 2,Table 3 & Table 4,we observe that sum of absolute error is very less in analytical solution compare to numerical solution.we get the result obtained by analytical methods is always providing accurate solution but numerical solution always providing approximate result. Since it is not possible to solve every partial differential equation analytically so numerical methods providing a good agreement in those cases where solutions not exist or we are unable to solve partial differential equation analytically. This comparative study provides better result for numerical solution of one dimensional heat equation.

## References:

1. Abhulimen CE, Omowo BJ. Modified Crank-Nicolson method for solving one-dimensional parabolic equation. International Journal of Scientific Research. 2019;15(6):60–66.
2. Agyeman E, Folson D. A comparison of numerical methods for solving partial differential equations. International Journal of Computer Applications. 2013;79(5):11.
3. Eves H. Great moments in mathematics (Vol. 1: Before 1650; Vol. 2: After 1650). The Mathematical Association for America, Washington DC; c1983.
4. Faddeeva VN. Computational methods in linear algebra. Dover, New York;c 1959.
5. Farin G. Curves and surfaces for computer-aided design: A practical guide. 2nd ed. Academic Press, Boston; c1990.
6. Fehlberg E. Classical fifth, sixth, seventh and eighth order Runge-Kutta formulas with stepsize control. NASA TR R-287; c1968.
7. Finizio N, Ladas G. An introduction to differential equations with difference equations, Fourier series, and partial differential equations. Wadsworth Publishing, Belmont, CA; c1982.
8. Fausett LV. Applied numerical analysis using MATLAB. Texas A&M University-Commerce; c2009.
9. Aubert G, Kornprobst P. Mathematical problems in image processing: Partial differential equations and the calculus of variations. 2nd ed. Springer, New York, NY; c2006.
10. Chen G, Zhou J. Boundary element methods. Academic Press, New York, NY; c1992.

11. Grewal BS. Numerical methods in engineering and science: With programs in C and C++. Khanna Publishers, Nai Sarak, Delhi; c2013.
12. Versteeg H, Malalasekra W. An introduction to computational fluid dynamics: The finite volume method. 2nd ed. Prentice Hall, Upper Saddle River, NJ; c2007.
13. Karatzas I, Shreve SE. Methods of mathematical finance. Springer, New York, NY; c1998.
14. Liu T. A study on hybrid numerical techniques for solving fluid dynamics problems. International Journal of Hybrid Information Technology. 2015;8(8):91.
15. Rao KS. Numerical methods for scientists and engineers. 3rd ed. PHI Learning Private Ltd., New Delhi; c2006.
16. Robinson JC. Infinite-dimensional dynamical systems: An introduction to dissipative parabolic PDEs and the theory of global attractors. Cambridge University Press, Cambridge; c2001.