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A study on fuzzy representations and algebraic structures

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Abstract
Fuzzy representations and algebraic structures are basic mathematical frameworks that are used to explain uncertainty and complexity in different systems. This paper explores the complex interactions between algebraic structures and fuzzy representations, including their applications, theoretical foundations, and mutual benefits. We start by giving a general introduction to fuzzy sets and how they are represented. Then, we look at algebraic structures like groups, rings, and lattices. Finally, we explore how these two areas work together. We want to clarify the significant effects of fuzzy representations on algebraic structures and vice versa via this research, highlighting their importance in a variety of domains such as control systems, artificial intelligence, and decision-making.

Keywords: Fuzzy sets, fuzzy representations, algebraic structures

Introduction
The foundation of mathematical modelling is made up of fuzzy representations and algebraic structures, which provide strong tools for addressing the ambiguity and uncertainty prevalent in real-world situations. Lotfi Zadeh's invention of fuzzy set theory in the 1960s transformed our understanding of and ability to evaluate imprecise data, and algebraic structures provide a formal foundation for the study of abstract mathematical systems. In order to clarify their theoretical underpinnings, real-world applications, and reciprocal impacts, this article will examine the complex link between fuzzy representations and algebraic structures [1].

Fuzzy sets are fundamental to fuzzy logic because they are a generalisation of classical sets that take membership degrees into account. Fuzzy sets, which represent the inherent ambiguity in many real-world settings, permit incremental membership, in contrast to crisp sets, where an element either belongs or does not. With a broad palette of mathematical tools for reasoning under uncertainty, fuzzy representations provide a formal foundation for expressing and manipulating ambiguous or incomplete data. Fuzzy logic operations allow for flexible manipulation of fuzzy sets, making it easier to describe complicated systems with ambiguous or uncertain inputs. Examples of these operations include fuzzy intersection, fuzzy union, and fuzzy complement.

Parallel to this, algebraic structures are the fundamental building blocks of abstract algebra and include a wide range of mathematical entities, including fields, rings, groups, and lattices. As the quintessential algebraic structures, groups are made up of a set and a binary operation that satisfies closure, associativity, identity, and invertibility. They are used extensively in many different domains. Subject to certain axioms, rings add another binary operation-usually addition and multiplication to the concept of groups. In the meanwhile, lattices provide a framework for researching ordered structures by encapsulating supremum/infimum operations and interactions involving partial ordering [1].

Fuzzy representations and algebraic structures have many mutually beneficial relationships; one field greatly enhances the other. An intuitive way to comprehend fuzzy logic operations is inside algebraic structures, which serves as a link between abstract algebra and imprecise thinking. Fuzzy complementation, for example, is similar to the concept of complement in Boolean algebra, whereas fuzzy intersection and union procedures are comparable to the meet and join operations in lattice theory. Furthermore, algebraic structures allow for rigorous reasoning about fuzzy systems by providing a formal vocabulary to analyse the characteristics and interactions of fuzzy sets [1].

When fuzzy representations and algebraic structures are combined, a potent toolset for solving a variety of real-world issues is produced. Fuzzy logic in artificial intelligence offers an adaptable framework for simulating human-like thought and decision-making processes, allowing intelligent computers to efficiently manage ambiguous or lacking information. Fuzzy control methods are used by control systems to handle complicated nonlinear systems, providing reliable solutions in situations when exact mathematical models are not accessible. Moreover, the combination of fuzzy representations and algebraic structures finds uses in a variety of domains, including as data analysis, pattern recognition, and optimisation, demonstrating its adaptability and usefulness in tackling challenging, real-world problems [2].

Fuzzy sets

The fuzzy set hypothesis was first presented in the journal Information and Control in 1965 with a work by L.A. Zadeh titled Fuzzy Sets. This paper serves as a foundation for the development of the novel mathematical theory. In his work, Zadeh extended the conventional understanding of the Cantor set by allowing the membership capacity to take values 0 and 1, in addition to any incentive from intermediate [0,1]. They're referred to as "fuzzy" sets. When J.A. Goguen introduced the concept of the L-set in 1968, he improved upon Zadeh's ideas. He allowed the membership capability to take esteems on a global lattice L in addition to inner [0, 1]. Mathematicians and experts alike were enthralled by the concept of fuzzy sets, which linked mathematical concepts, ideas, and outcomes to illustrate many real-world processes. There are a number of applications for fuzzy sets that we might list, including fundamental leadership, careful calculation, problem-solving, control hypothesis, design acknowledgment, image preparation, and many more [3].

The following is a possible characterization of mathematically fuzzy sets:

Definition 2

A mapping μ: X → [0, 1] characterises a fuzzy set μ in the universe X (a fuzzy subset of X). X's fuzzy subset arrangements are denoted by [0, 1] X. A special case of fuzzy sets are crisp sets. A crisp set's participation capacity is measured on the lattice L = {0, 1}. It suggests that if a point x ∈ X has a place or does not have a place with μ independently, it might be assigned a number, either 1 or 0. Using a t-standard T and a t-conorm S characterises operations on fuzzy sets [4]. The next section will illustrate these concepts.

Definition 3

A fuzzy set μ∩ν is used to characterise a crossing point of fuzzy sets μ and ν, with the ultimate objective being

$$(\mu \cap \nu)(x) = T\,(\mu(x), \nu(x)).$$

Definition 4

A fuzzy set μ∪ν is defined as a union μ∪ν of fuzzy sets μ and ν, with the ultimate objective that

$$(\mu \cup \nu)(x) = S\,(\mu(x), \nu(x)).$$

An involution characterises a correlative organisation of a fuzzy collection:

Definition 5

In the unlikely event that a capacity N: [0, 1] → [0, 1] satisfies the subsequent constraints for every x, y ∈ [0, 1], it is referred to as an order turning around involution:

- $N(N(x)) = x$
- $N(x) \geq N(y)$ whenever $x \leq y$.

Definition 6

A fuzzy set μc that is a supplement of a fuzzy set μ is defined so that μ c (x) = N (μ(x)) [5].

In the event that lattice L = [0, 1] should occur, a fuzzy set μ supplement is often described as

$$\mu\, c\,(x) = 1 - x.$$

The concepts of a fuzzy number and a fuzzy interim play a crucial role in fuzzy mathematics [6]. These fuzzy sets are rare varieties in which the area is a subset of the true line. Interims and fuzzy numbers are often used in applications as well as speculative research. Several approaches are available for characterising fuzzy numbers. We might use trapezoidal fuzzy numbers as an example of a widely used fuzzy number [7]:

$$\mu(x) = \begin{cases} 0, \; x \leq a, \\ \frac{x-a}{b-a}, \; a \leq x \leq b, \\ 1, \; b \leq x \leq c, \\ \frac{x-d}{c-d}, \; c \leq x \leq d, \\ 0, \; d \leq x, \end{cases}$$

In the case when b = c, we get a triangular fluffy number with $a < b \leq c < d$ [8].

In the thesis, we examine fluffy real numbers as defined by B. Hutton and then examined by other authors [9].

The fuzzy operators

To meet the special membership features of fuzzy logic for values totally in the area of 0 and 1, we are rewriting the administrators of the classical set hypothesis with the explicit purpose of effectively controlling fuzzy sets. In contrast to the consistently identical meanings of the attributes of fuzzy sets, the meaning of administrators on fuzzy sets is chosen, much like membership capacity. These are the two administrator configurations that are most often used for the complement (NOT), intersection (AND), and union (OR) [10]:

Name	Intersection and: 14/MB(1)	Union OU: pAuil(x)	Complement NOT: µ%i(x)
Zadeh Operators MIN/MAX	min (PA (x), ps(x))	max (PAK ;LBW)	1 — pA(x)
Probabilistic PROD/PROBOR	PA(x) x AB(x) _	litt(x) + iis(x) — PAW x liB(x)	1⁻PALO

Using fuzzy adminidstrators' common definitions, we may often find the traits of commutativity, distributivity, and associativity in artistic works. However, there are two exceptional rare instances [11-13]:

- The rule of banned centre is negated in fuzzy logic as follows: A ∧ A¯ 6= X, or µA∧A¯(x) 6= 1.
- In fuzzy reasoning, a component may now belong with A and not An: µA∩A¯(x) 6=0, or A ∩ A¯ 6=∅. Keep in mind that the set supp (A) – noy (A) is related to these components.

Fuzzy representations of fuzzy groups

Definition [14]

Let G be a group, T: G → GL (M) be a representation of G in M, and M be a vector space over K. Assume that ν is a fuzzy group on T (G) and that µ is a fuzzy group on G. If T is a fuzzy homomorphism of µ onto ν, then the representation T is a fuzzy representation.

Example

Let M be a vector space over R and let G = (Z, +).

Let $T : G \to GL(M)$ be defined by $T(x) = T_x$ where $T_x : M \to M,$ such that $T_x(m) = xm$, for $x \in G$ and $m \in M$. Then T serves as a metaphor.

Define μ on G now by

$$\mu(x) = \begin{cases} 1, & \text{if } x \text{ is even} \\ 1/2, & \text{if } x \text{ is odd.} \end{cases}$$

Then, on G, μ is a fuzzy group.

Let ν be the fuzzy group defined by on the set of values of T.

$$\nu(T_{\text{even}}) = 1, \quad \nu(T_{\text{odd}}) = 1/2$$

Then, we have,

$$T(\mu)(T_{\text{even}}) = \vee\{\mu(x) | x \in T^{-1}(T_{\text{even}})\} = 1$$

Similarly,

$$T(\mu)(T_{\text{odd}}) = 1/2.$$
$$T(\mu) = \nu.$$

So, T is a hazy depiction of µ onto ν.

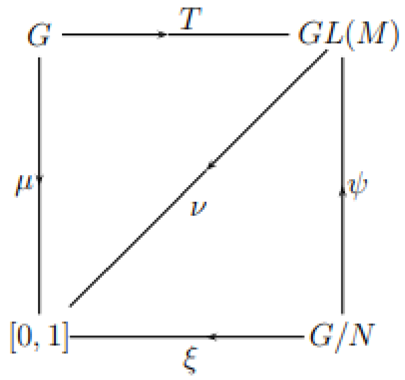
Theorem [15]

Let μ be a fuzzy group on G and let N be a normal subgroup of G. Define $\xi \in F(G/N)$ by $\xi([x]) = \vee\{\mu(z) | z \in [x]\}, \forall x \in G,$ where [x] denotes the coset xN. Then ξ is a fuzzy group on *G/N*.

Theorem (A fundamental theorem of fuzzy representations)

Let *G* be a group and *M* be a vector space over a field *K*. If T is a fuzzy representation of *G*, then $\psi : G/N \to GL(M)$ defined by $\psi([x]) = T(x), \quad x \in G,$ is a fuzzy representation of *G/N*, where *N* is a normal subgroup of *G*.

Proof. Let µ be a fuzzy group on *G*. Since *T* is a fuzzy representation, ∃ a fuzzy group *ν* on *T (G)* such that $T(\mu) = \nu$.
We have to prove that ψ is a fuzzy representation of *G*.



Given that $\psi : G/N \to GL(M)$ defined by $\psi([x]) = T(x) = T_x, \forall x \in G.$ Then ψ is a homomorphism of *G/N* into *GL (M).*

For $[x], [y] \in G/N,$

$$\begin{aligned} \psi([x][y]) &= \psi([xy]) = T_{xy}, \quad x, y \in G \\ T_{xy}(m) &= (xy)(m) = x(ym) \\ &= T_x(ym) \\ &= T_x(T_y(m)) \\ &= (T_xT_y)(m), \quad m \in M \\ \therefore\ T_{xy} &= T_xT_y \\ \therefore\ \psi([xy]) &= \psi([x])\psi([y]) \\ \psi([\alpha x]) &= T_{\alpha x} \\ T_{\alpha x}(m) &= (\alpha x)(m) \\ &= \alpha(xm) \\ &= \alpha T_x(m) \\ &= (\alpha T_x)(m), \quad m \in M, \alpha \in K \end{aligned}$$

$$\therefore\ T_{\alpha x} = \alpha T_x$$

$$\therefore\ \psi([\alpha x]) = \alpha\psi([x]).$$

Hence $\psi : G/N \to GL(M)$ is a representation.

For $x \in G, \exists$ an element $xN = [x] \in G/N.$

$$\psi(\xi)(y) = \vee\{\xi([x]) | [x] \in \psi^{-1}(y),\ y \in \psi(G/N)\}$$
$$= \vee\{\vee\{\mu(z)|z \in [x], x \in G\}, y \in T(G)\}$$
$$= \vee\{\mu(z)|z \in [x], x \in G,\ y \in T(G)\}$$
$$= T(\mu)(y)$$
$$\therefore\ \psi(\xi) = T(\mu) = \nu.$$

∴ ψ is a fuzzy representation of ξ onto ν.

Example

Let $G = \{1, -1, i, -i\}$, a group under usual multiplication and *M* be a vector space over *R*.

Let $N = \{1, -1\}$. Then N is a normal subgroup of G.

Let $T : G \to GL(M)$ be defined by $T(x) = T_x$, where $T_x(M) = xm,\ \forall\ x \in G$ and $m \in M.$

Define μ on G by

$$\mu(x) = \begin{cases} 1, & \text{when } x \text{ is } 1 \text{ or } -1 \\ 0.7, & \text{when } x \text{ is } i \text{ or } -i. \end{cases}$$

Then μ is a fuzzy group on G.

Let *ν* be a fuzzy group on $T(G)$, defined by $\nu(T_1) = 1, \nu(T_{-1}) = 1,$

$\nu(T_i) = 0.7$ and $\nu(T_{-i}) = 0.7.$ Then

$$T(\mu)(T_1) = \vee\{\mu(x)/x \in T^{-1}(T_1)\} = 1$$

Similarly we get,

$$T(\mu)(T_{-1}) = 1,\quad T(\mu)T(i) = 1/2,\quad T(\mu)(T_{-i}) = 1/2$$
$$\therefore\ T(\mu) = \nu$$

∴ T is a fuzzy representation of μ onto ν.

$$G/N = \{N, -N, iN, -iN\}$$
$$= \{N, iN\}$$
$$\psi(\xi)(T_1) = \vee\{\xi[1] | [1] \in \psi^{-1}(T_1), T_1 \in \psi(G/N)\}$$
$$= \vee\{\vee\{\mu(z)|z \in [1], 1 \in G\}, T_1 \in T(G)\}$$
$$= \vee\{\mu(z)|z \in [1], 1 \in G, T_1 \in T(G)\}$$
$$= 1$$
$$\psi(\xi)(T_{-1}) = 1,\quad \psi(\xi)(T_i) = 0.7,\quad \psi(\xi)(T_{-i}) = 0.7.$$

$\therefore \psi(\xi) = \nu.$ Hence ψ is a fuzzy representation of ξ onto ν.

Example

Let $G = (R - \{0\}, \times)$ and $N = \{1, -1\}.$ Define $T : G \to GL(M)$ by $T(x) = T_x, \forall\ x \in G, m \in M,$ where M is a vector

Space over K.
\Define μ on G by

$$\mu(x) = \begin{cases} t, & \text{when } x = 1 \text{ or } -1 \\ r, & \text{when } x \in Q - \{1, -1, 0\}. \\ s, & \text{when } x \in R - Q. \end{cases}$$

Where, $t, r, s \in [0, 1]$ and $t > r > s.$

Then μ is a fuzzy group on G. Let ν be the fuzzy group on T (G) defined by

$\nu(T_1) = t, \nu(T_{-1}) = t, \nu(T_i) = r, i \in Q - \{1, -1, 0\}$ and $\nu(T_j) = s,\ j \in R - Q.$ Then

$$T(\mu)(T_1) = \vee\{\mu(x)|x \in T^{-1}(T_1)\} = t$$

Similarly,

$$T(\mu)(T_{-1}) = t,\quad T(\mu)(T_i) = r, T(\mu)(T_j) = s.$$

$\therefore T(\mu) = \nu.$ Hence T is a fuzzy representation of μ onto ν.

Given that

$$\psi : G/N \to GL(M) \text{ by } \psi([x]) = T_x = T(x), \forall\ x \in G.$$

Then ψ is a representation.

$$G/N = \{N, -N, iN, jN\}, i \in Q - \{1, -1, 0\} \text{ and } j \in R - Q$$
$$= \{N, iN, jN\}.$$

Define

$$\xi \text{ on } G/N \text{ by } \xi(N) = t, \xi(iN) = r, \xi(jN) = s.$$

Then ξ is a fuzzy group on G/N.

$$\psi(\xi)(T_1) = \vee\{\xi[1] | [1] \in \psi^{-1}(T_1)\}, T_1 \in \psi(G/N)\}$$
$$= \vee\{\xi(N) | \psi(N) = T_1, T_1 \in T(G)\}$$
$$= t.$$

$$\psi(\xi)(T_{-1}) = \vee\{\xi[-1] | [-1] \in \psi^{-1}(T_{-1})\}, T_{-1} \in T(G)\}$$
$$= \vee\{\xi(N) | N \in \psi^{-1}(T_{-1}), T_{-1} \in T(G)\}$$
$$= t.$$

For $i \in Q - \{1, -1, 0\}, \psi(\xi)(T_i) = r.$

For $j \in Q', \psi(\xi)(T_j) = s.$

Hence $\psi(\xi) = \nu.$ Thus, ψ is a fuzzy representation of ξ onto ν.

Fuzzy algebraic structures

Definition [15]

Let μ be a fuzzy group on a group G. Given x ∊ G, the least positive integer 'n' such that μ(x n) = μ(e) is called the fuzzy order of x with respect to μ. If no such 'n' exists, x is said to have infinite fuzzy order with respect to μ. The fuzzy order of x with respect to μ is denoted by FOμ(x)

Example

Let G and μ be as in example 1.4.4. Then

$$\mu((-1)^2) = \mu(1) \ \therefore FO_\mu(-1) = 2$$
$$\mu(i^4) = \mu(1) \ \therefore FO_\mu(i) = 4$$
$$\mu((-i)^4) = \mu(1) \ \therefore FO_\mu(-i) = 4.$$

Example

Let G = (R − {0}, ×). Define μ on G by

$$\mu(x) = \begin{cases} 1, & \text{when } x = 1 \\ 0.7, & \text{when } x = -1 \\ 0.3, & \text{when } x \in Q - \{1, -1\} \\ 0.1, & \text{when } x \in R - Q. \end{cases}$$

Then μ is a fuzzy group on G.

$\mu((-1)^2) = \mu(1), \quad \therefore FO_\mu(-1) = 2$

But for any $x \in R - \{1, 0, -1\}$, $\nexists$ a (+)ve integer n such that $\mu(x^n) = \mu(1)$.

$$\therefore FO_\mu(x) = \infty, \quad \forall x \in R - \{1, 0, -1\}.$$

Conclusion

In conclusion, the study of fuzzy representations and algebraic structures unveils a fascinating interplay between uncertainty modelling and abstract algebra, offering powerful tools for reasoning under uncertainty and complexity. By bridging the gap between imprecise reasoning and rigorous mathematics, this synergy opens up new avenues for tackling real-world problems in diverse domains. As we continue to unravel the intricate relationships between fuzzy representations and algebraic structures, we pave the way for innovation and advancement in fields ranging from artificial intelligence to control systems, propelling the frontier of mathematical modelling into new realms of possibility.

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