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Fusion classes of Non-Abelian Metabelian groups of order upto 24

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Abstract
In this paper G be the non-abelian metabelian group and denotes the fusion class of element a in G. Fusion class is an equivalence relation. A group is metabelian group if its commutator group is abelian or if it has an abelian normal subgroup in which the factor group is also abelian. There are 35 non-Abelian metabelian groups of order less than or equal to 24. In this paper, we investigate the fusion classes of all non-abelian metabelian groups of order less than or equal to 24 and the verification has been made through GAP(Groups Algorithm Programming) software.

Keywords: Metabelian groups, automorphism, fusion classes

Introduction

Let $\boldsymbol{G}$ be a finite metabelian group, $\boldsymbol{Z_n}$ denotes the cyclic group of order$\boldsymbol{n}$, $\boldsymbol{S_n}$ denotes the permutation group of degree $\boldsymbol{n}$, $\boldsymbol{D_n}$ denotes the dihedral group of order$\boldsymbol{2n}$, $\boldsymbol{Q_n}$ denotes quaternion group. A group is said to be metabelian if $\boldsymbol{G'}$, the derived subgroup of $\boldsymbol{G}$ is abelian. Much has been investigated about the properties of metabelian groups in the literature. In paper [3], structure of metabelian groups of order upto $\mathbf{24}$ has been described. In paper [4], the authors studied about the conjugacy classes of metabelian groups of order less tha$\mathbf{24}$. In paper [1, 2], automorphisms of some non-abelian groups of order $\boldsymbol{p^4}$are computed. In the present paper, we shall find the fusion classes of metabelian non-abelian groups of order less than or equal to $\mathbf{24}$.
In [3], Rehman Abdul described the metabelian groups of order less than or equal to24.

(1) $D_3 \cong S_3 \cong < a, b; a^3 = b^2 = 1, b\ a\ b = a^{-1} >$.

(2)$D_4 \cong < a, b; a^4 = b^2 = 1, b\ a\ b = a^{-1} >$.

(3)$Q_3 \cong < a, b: a^4 = 1, b^2 = a^2, a\ b\ a = b >$.

(4)$D_5 \cong < a, b: a^5 = b^2 = 1, b\ a\ b = a^{-1} >$.

(5)$Z_3 \rtimes Z_4 \cong < a, b: a^3 = b^4 = 1, b^{-1} a\ b = a^2 >$.

(6)$A_4 \cong < a, b, c: a^2 = b^2 = c^3 = 1, b\ a = a\ b, c\ a = a\ b\ c, c\ b = a\ c >$.

(7)$D_6 \cong < a, b: a^6 = b^2 = 1, b\ a\ b = a^{-1} >$.

(8)$D_7 \cong < a, b: a^7 = b^2 = 1, b\ a\ b = a^{-1} >$.

(9)$D_8 \cong < a, b: a^8 = b^2 = 1, b\ a\ b = a^{-1} >$.

(10)$G \cong < a, b: a^8 = b^2 = 1, b\ a\ b = a^3 >$.
(11)$Q_{16} \cong < a, b: a^8 = 1, \quad a^4 = b^2, \quad a\ b\ a = b >$.

(12)$D_4 \times Z_2 \cong < a, b, c: a^4 = b^2 = c^2 = 1, a\ c = c\ a, b\ c = c\ b, b\ a\ b = a^{-1} >$.

(13)$Q_3 \times Z_2 \cong < a, b, c: a^4 = b^4 = c^2 = 1, b^2 = a^2, b\ a = a^3 b, a\ c = c\ a, b\ c = c\ b >$.

$(14) Modular - 16 = G \cong < a, b: a^8 = b^2 = 1, a\ b = b\ a^5 >.$

$(15) B \cong < a, b: a^4 = b^4 = 1, a\ b = b\ a^3 >.$

$(16) K \cong < a, b, c: a^4 = b^2 = c^2 = 1, a\ b = b\ a, a\ c = c\ a, c\ b = a^2 b\ c >.$

$(17) G_{4,4} \cong < a, b: a^4 = b^4 = (a\ b\ )^2 = 1, a\ b^3 = b\ a^3 >.$

$(18) D_9 \cong < a, b: a^9 = b^2 = 1, b\ a\ b = a^{-1} >.$

$(19) S_3 \times Z_3 \cong < a, b, c: a^3 = b^2 = c^3 = 1, b\ a = a^{-1} b, a\ c = c\ a, b\ c = c\ b >.$

$(20) (Z_3 \times Z_3) \rtimes Z_2 \cong < a, b, c: a^2 = b^3 = c^3 = 1, b\ c = c\ b, b\ a\ b = a, c\ a\ c = a >.$

$(21) D_{10} \cong < a, b: a^{10} = b^2 = 1, b\ a\ b = a^{-1} >.$

$(22) Fr_{20} \cong Z_5 \rtimes Z_4 \cong < a, b: a^4 = b^5 = 1, b\ a = a\ b^2 >.$

$(23) Z_5 \rtimes Z_4 \cong < a, b: a^4 = b^5 = 1, b\ a\ b = a >.$

$(24) Fr_{21} \cong Z_7 \rtimes Z_3 \cong < a, b: a^3 = b^7 = 1, b\ a = a\ b^2 >.$

$(25) D_{11} \cong < a, b: a^{11} = b^2 = 1, b\ a\ b = a^{-1} >.$

$(26) S_3 \times Z_4 \cong < a, b, c: a^3 = b^2 = c^4 = 1, a^b = a^{-1}, a\ c = c\ a, b\ c = c\ b >.$

$(27) S_3 \times Z_2 \times Z_2 \cong < a, b, c, d: a^3 = b^2 = c^2 = d^2 = 1, a^b = a^{-1}, a\ c = c\ a, a\ d = d\ a, b\ c = c\ b, b\ d = d\ b, c\ d = d\ c >.$

$(28) D_4 \times Z_3 \cong < a, b, c: a^3 = b^4 = c^2 = 1, b^c = b^{-1}, a\ b = b\ a, a\ c = c\ a >.$

$(29) Q \times Z_3 \cong < a, b, c: a^4 = c^3 = 1, a^2 = b^2, a^b = a^{-1}, b\ c = c\ b, a\ c = c\ a >.$

$(30) A_4 \times Z_2 \cong < a, b, c, d: a^2 = b^2 = c^3 = d^2 = 1, a\ b = b\ a, c\ a = a\ b\ c, a\ d = d\ a, a\ c = c\ b, b\ d = d\ b, c\ d = d\ c >.$

$(31) Q_{12} \cong < a, b: a^{12} = 1, a^6 = b^2, b^{-1} a\ b = a^{-1} >.$

$(32) D_{12} \cong < a, b: a^{12} = b^2 = 1, b\ a\ b = a^{-1} >.$

$(33) Z_6 \rtimes Z_4 \cong < a, b: a^4 = b^6 = 1, b\ a\ b = a >.$

$(34)\ Z_3 \rtimes Z_8 \cong < a, b: a^3 = b^8 = 1, b\ a\ b^{-1} = a^{-1} >.$

$(35)\ Z_3 \rtimes Q \cong < a, b, c: a^2 = b^6 = c^2 = 1, ab = ba, \quad ac = ca, (cb\ )^2 = 1 >.$

Fusion classes of all non-abelian metabelian groups of order less than equal to 24
Fusion classes of groups of order 6
Fusion Classes of *D3*

$D_3 \cong S_3 \cong < a, b; a^3 = b^2 = 1, bab = a^{-1} >$
Fusion Classes are

$\overline{cl(1)} = \{1\}$

$\overline{cl(a)} = \{a, a^2\}$

$\overline{cl(b)} = \{ab, a^2b, b\}$

So there are 3 fusion classes in D_3.

Fusion Classes of groups of order 8
Fusion classes of D_4

$D_4 \cong< a, b; a^4 = b^2 = 1, bab = a^{-1} >$

Fusion Classes are

$\overline{cl(1)} = \{1\}$

$\overline{cl(a)} = \{a^i; (i, 2) = 1,1 \leq i < 4\}$

$\overline{cl(a^2)} = \{a^2\}$

$\overline{cl(b)} = \{a^ib; 1 \leq i \leq 4\}$

Therefore, there are 4 fusion classes in D_4.

Fusion Classes of Q_3

$Q_3 \cong< a, b: a^4 = 1, b^2 = a^2, aba = b >$

Thus, the fusion classes are

$\overline{cl(1)} = \{1\}$

$\overline{cl(a)} = \{a, b, a^3, b^3, ab, a^3b\}$

$\overline{cl(a^2)} = \{a^2\}$

Hence the total number of fusion classes are 3
Fusion Classes of groups of order 10
Fusion Classes of D_5

$D_5 \cong< a, b: a^5 = b^2 = 1, bab = a^{-1} >$

Fusion Classes are

$\overline{cl(1)} = \{1\},$

$\overline{cl(a)} = \{a^i; (i, 5) = 1\},$

$\overline{cl(b)} = \{a^ib; 1 \leq i \leq 5\}.$

So there are 3 fusion classes
Fusion Classes of groups of order 12
Fusion Classes of $Z_3 \rtimes Z_4$

$Z_3 \rtimes Z_4 \cong< a, b: a^3 = b^4 = 1, b^{-1}ab = a^2 >$

Fusion Classes are:

$\overline{cl(1)} = \{1\},$

$\overline{cl(a)} = \{a^i; 1 \le i < 3\}$,

$\overline{cl(b)} = \{a^i b^j; 2 \nmid j, 0 \le i \le 2, 1 \le j \le 3\}$,

$\overline{cl(b^2)} = \{b^2\}$,

$\overline{cl(ab^2)} = \{a^i b^2; 1 \le i \le 2\}$.

Hence there are 5 fusion classes
Fusion Classes of A_4

$A_4 \cong \langle a, b, c : a^2 = b^2 = c^3 = 1, ba = ab, ca = abc, cb = ac \rangle$

Fusion Classes are:

$\overline{cl(1)} = \{1\}$,

$\overline{cl(a)} = \{a^i b^j; 0 \le i, j \le 1, \quad i + j \ne 0\}$,

$\overline{cl(c)} = \left\{a^i b^j c^k; 0 \le i, j \le 1, \quad k \in \{1,2\}\right\}$.

Thus, total number of fusion classes are 3
Fusion Classes of D_6

$D_6 \cong \langle a, b : a^6 = b^2 = 1, bab = a^{-1} \rangle$

Fusion Classes are

$\overline{cl(1)} = \{1\}$,

$\overline{cl(a)} = \left\{a^i; \quad i \in \{1,5\}\right\}$,

$\overline{cl(a^2)} = \left\{a^i; \quad i \in \{2,4\}\right\}$,

$\overline{cl(a^3)} = \{a^3\}$,

$\overline{cl(b)} = \{a^i b; 0 \le i \le 5\}$.

Total number of fusion classes are 5
Fusion Classes of groups of order 14
Fusion Classes of D_7

$D_7 = \langle a, b : a^7 = b^2 = 1, bab = a^{-1} \rangle$

Fusion Classes are

$\overline{cl(1)} = \{1\}$,

$\overline{cl(a)} = \{a^i; 1 \le i \le 6\}$,

$\overline{cl(b)} = \{a^i b; 0 \le i \le 6\}$.

So there are 3 fusion classes
Fusion Classes of groups of order 16
Fusion classes of D_8
Fusion Classes are

$$\overline{cl(1)} = \{1\},$$

$$\overline{cl(a)} = \{a^i; (i,2) = 1, 1 \le i < 8\},$$

$$\overline{cl(a^2)} = \{a^2, a^6\},$$

$$\overline{cl(a^4)} = \{a^4\},$$

$$\overline{cl(b)} = a^i b; 0 \le i \le 7.$$

Total number of fusion classes are 5
Fusion Classes of G

$$G \cong < a, b : a^8 = b^2 = 1, bab = a^3 >$$

Fusion Classes are

$$\overline{cl(1)} = \{1\},$$

$$\overline{cl(a)} = \{a^i; (2,i) = 1, 1 \le i \le 8\},$$

$$\overline{cl(a^2)} = \{a^{2i}; (2,i) = 1, 1 \le i \le 4\},$$

$$\overline{cl(a^4)} = \{a^4\},$$

$$\overline{cl(b)} = \left\{a^i b; \frac{2}{i}, 0 \le i < 8\right\},$$

$$\overline{cl(ab)} = \{a^i b; (2,i) = 1, 1 \le i \le 8\}.$$

So there are 6 fusion classes
Fusion Classes of Q_{16}

$$Q_{16} \cong < a, b : a^8 = 1, a^4 = b^2, \quad aba = b >$$

Fusion Classes are

$$\overline{cl(1)} = \{1\},$$

$$\overline{cl(a)} = \{a^i; (2,i) = 1, 1 \le i \le 8\},$$

$$\overline{cl(a^2)} = \{a^2, a^6\},$$

$$\overline{cl(a^4)} = \{a^4\},$$

$$\overline{cl(b)} = \{a^i b^j; 0 \le i \le 3, \quad j = 1,3\}.$$

Therefore, there are 5 fusion classes

Fusion classes of $D_4 \times Z_2$

$$D_4 \times Z_2 \cong < a, b, c: a^4 = b^2 = c^2 = 1, ac = ca, bc = cb, bab = a^{-1} >$$

Fusion classes are

$$\overline{cl(1)} = \{1\},$$

$$\overline{cl(a)} = \left\{a^i c^j; \quad i \in \{1,3\}, \quad j \in \{0,1\}\right\},$$

$$\overline{cl(b)} = \left\{a^k b c^l; 0 \leq k \leq 3, \quad l \in \{0,1\}\right\},$$

$$\overline{cl(c)} = \{c, a^2 c\},$$

$$\overline{cl(a^2)} = \{a^2\}.$$

Therefore, there are 5 **fusion classes**

Fusion classes of $Q_3 \times Z_2$

$$Q_3 \times Z_2 \cong < a, b, c: a^4 = b^4 = c^2 = 1, b^2 = a^2, ba = a^3 b, ac = ca, bc = cb >$$

\Fusion classes are

$$\overline{cl(1)} = \{1\},$$

$$\overline{cl(a)} = \{a, a^3, b, b^3, ab, ac, a^3 b, a^3 c, bc, abc, a^2 bc, a^3 bc\},$$

$$\overline{cl(a^2)} = \{a^2\},$$

$$\overline{cl(c)} = \{c, a^2 c\}.$$

Total number of fusion classes are 4

Fusion classes of $Modular - 16$

$$Modular - 16 = G \cong < a, b: a^8 = b^2 = 1, ab = ba^5 >$$

Fusion classes are

$$\overline{cl(1)} = \{1\},$$

$$\overline{cl(a)} = \left\{a^i b^j; 2 \nmid i, 1 \leq i \leq 8, \quad j \in \{0,1\}\right\},$$

$$\overline{cl(b)} = \{b, a^4 b\},$$

$$\overline{cl(a^2)} = \{a^2, a^6\},$$

$$\overline{cl(a^2 b)} = \{a^2 b, a^6 b\},$$

$$\overline{cl(a^4)} = \{a^4\}.$$

Total number of fusion classes are 6

Fusion classes of B

$$B \cong < a, b: a^4 = b^4 = 1, ab = ba^3 >$$

Fusion classes are

$\overline{cl(1)} = \{1\},$

$\overline{cl(a)} = \left\{a^i b^j; \quad i \in \{1,3\}, \quad j \in \{0,2\}\right\},$

$\overline{cl(b)} = \left\{a^l b^m; 0 \leq l \leq 3, \quad m \in \{1,3\}\right\},$

$\overline{cl(a^2)} = \{a^2\},$

$\overline{cl(b^2)} = \{b^2\},$

$\overline{cl(a^2b^2)} = \{a^2 b^2\}.$

Total number of fusion classes are 6
Fusion classes of $\boldsymbol{K}$

$K \cong < a, b, c: a^4 = b^2 = c^2 = 1, ab = ba, ac = ca, cb = a^2 bc >$

Fusion classes are

$\overline{cl(1)} = \{1\},$

$\overline{cl(a)} = \left\{a^i; \quad i \in \{1,3\}\right\},$

$\overline{cl(b)} = \{b, c, a^2 b, a^2 c, abc, a^3 bc\},$

$\overline{cl(a^2)} = \{a^2\},$

$\overline{cl(ab)} = \{ab, ac, bc, a^3 b, a^3 c, a^2 bc\}.$

Total number of fusion classes are 5
Fusion classes of $\boldsymbol{G_{4,4}}$

$G_{4,4} \cong < a, b: a^4 = b^4 = (ab)^2 = 1, ab^3 = ba^3 >$

Fusion classes are

$\overline{cl(1)} = \{1\},$

$\overline{cl(a)} = \{a, a^3, b, b^3, ab^2, a^2 b, a^2 b^3, a^3 b^2\},$

$\overline{cl(a^2)} = \{a^2, b^2\},$

$\overline{cl(ab)} = \{ab, a^3 b, ab^3, a^3 b^3\},$
$\overline{cl(a^2b^2)} = \{a^2 b^2\}.$

Total number of fusion classes are 5
Fusion Classes of groups of order $\boldsymbol{18}$
Fusion classes of $\boldsymbol{D_9}$

$D_9 \cong < a, b: a^9 = b^2 = 1, bab = a^{-1} >$

Fusion Classes are

$\overline{cl(1)} = \{1\},$

$\overline{cl(a)} = \{a^i; 3 \nmid i, 1 \leq i \leq 8\},$

$\overline{cl(a^3)} = \{a^i; \quad i \in \{3,6\}\},$

$\overline{cl(b)} = \{a^i b; 0 \leq i \leq 8\}.$

Total number of fusion classes are 4
Fusion classes of $S_3 \times Z_3$

$S_3 \times Z_3 \cong < a, b, c: a^3 = b^2 = c^3 = 1, ba = a^{-1}b, ac = ca, bc = cb >$

Fusion classes are

$\overline{cl(1)} = \{1\},$

$\overline{cl(a)} = \{a^j b; 0 \leq j \leq 2\},$

$\overline{cl(b)} = \{a^j b; 0 \leq j \leq 2\},$

$\overline{cl(c)} = \{c^k; 1 \leq k \leq 2\},$

$\overline{cl(ac)} = \{a^i c^j; 1 \leq i, j \leq 2\},$

$\overline{cl(bc)} = \{a^i bc^j; 0 \leq i \leq 2, 1 \leq j \leq 2\}.$

Total number of fusion classes are 6
Fusion classes of $(Z_3 \times Z_3) \rtimes Z_2$

$(Z_3 \times Z_3) \rtimes Z_2 \cong < a, b, c: a^2 = b^3 = c^3 = 1, bc = cb, bab = a, cac = a >$

Fusion classes are

$\overline{cl(1)} = \{1\},$

$\overline{cl(a)} = \{ab^i c^j; 0 \leq i, j \leq 2\},$

$\overline{cl(b)} = \{b^k c^l; 0 \leq k, l \leq 2, \quad k + l \neq 0\}.$

Total number of fusion classes are 3
Fusion Classes of groups of order 20
Fusion classes of D_{10}:

$D_{10} \cong < a, b: a^{10} = b^2 = 1, bab = a^{-1} >$

Fusion Classes are

$\overline{cl(1)} = \{1\},$

$\overline{cl(a)} = \{a^i; 2 \nmid i, 5 \nmid i, 1 \leq i < 10\},$

$\overline{cl(a^2)} = \{a^i; 2/i, 1 \leq i < 10\},$

$\overline{cl(a^5)} = \{a^5\},$

$\overline{cl(b)} = \{a^i b; 0 \leq i \leq 9\}.$

Total number of fusion classes are 5.

Fusion classes of Fr_{20}

$Fr_{20} \cong Z_5 \rtimes Z_4 \cong < a, b : a^4 = b^5 = 1, ba = ab^2 >$

Fusion classes are

$\overline{cl(1)} = \{1\},$

$\overline{cl(a)} = \{ab^i; 0 \leq i \leq 4\},$
$\overline{cl(b)} = \{b^j; 1 \leq j \leq 4\},$

$\overline{cl(a^2)} = \{a^2 b^i; 0 \leq i \leq 4\},$

$\overline{cl(a^3)} = \{a^3 b^i; 0 \leq i \leq 4\}.$

Total number of fusion classes are 5
Fusion classes of $Z_5 \rtimes Z_4$

$Z_5 \rtimes Z_4 \cong < a, b : a^4 = b^5 = 1, bab = a >$

Fusion classes are

$\overline{cl(1)} = \{1\},$

$\overline{cl(a)} = \{a^i b^j; \quad i \in \{1,3\}, 0 \leq j \leq 4\},$

$\overline{cl(b)} = \{b^j; 1 \leq j \leq 4\},$

$\overline{cl(a^2)} = \{a^2\},$

$\overline{cl(a^2 b)} = \{a^2 b^i; 1 \leq i \leq 4\}.$

Total number of fusion classes are5.
Fusion Classes of groups of order 21
Fusion classes of Fr_{21}

$Fr_{21} \cong Z_7 \rtimes Z_3 \cong < a, b : a^3 = b^7 = 1, ba = ab^2 >$

Fusion classes are

$\overline{cl(1)} = \{1\},$

$\overline{cl(a)} = \{ab^i; 0 \leq i \leq 6\},$

$\overline{cl(b)} = \{b^j; 1 \leq j \leq 6\},$

$\overline{cl(a^2)} = \{a^2b^i; 0 \le i \le 6\}$.

Total number of fusion classes are 4
Fusion Classes of groups of order 22
Fusion Classes of D_{11}

$D_{11} =< a, b: a^{11} = b^2 = 1, bab = a^{-1} >$

Fusion Classes are:

$\overline{cl(1)} = \{1\}$,

$\overline{cl(a)} = \{a^i; 1 \le i \le 10\}$,

$\overline{cl(b)} = \{a^i b; 0 \le i \le 10\}$.

So there are 3 fusion classes
Fusion classes of groups of order 24
Fusion classes of $S_3 \times Z_4$
$S_3 \times Z_4 \cong< x, y, z: x^3 = y^2 = z^4 = 1, x^y = x^{-1}, xz = zx, yz = zy >$

Fusion classes are

$\overline{cl(1)} = \{1\}$,

$\overline{cl(x)} = \{x^i; 1 \le i \le 2\}$,

$\overline{cl(y)} = \left\{x^j yz^k; 0 \le j \le 2, \quad k \in \{0,2\}\right\}$,

$\overline{cl(z)} = \{z^u; u \in \{1,3\}\}$,

$\overline{cl(xz)} = \left\{x^i z^j; 1 \le i \le 2, \quad j \in \{1,3\}\right\}$,

$\overline{cl(yz)} = \left\{x^i yz^k; 0 \le i \le 2, \quad k \in \{1,3\}\right\}$,

$\overline{cl(z^2)} = \{z^2\}$,

$\overline{cl(xz^2)} = \{x^i z^2; 1 \le i \le 2\}$.

Total number of fusion classes are 8

Fusion classes of $S_3 \times Z_2 \times Z_2$

$$S_3 \times Z_2 \times Z_2 \cong< x, y, z, w: x^3 = y^2 = z^2 = w^2 = 1, x^y = x^{-1}, xz = zx, xw = wx, yz = zy, yw = wy, zw = wz >$$

Fusion classes are

$\overline{cl(1)} = \{1\}$,

$\overline{cl(x)} = \{x^i; 1 \le i \le 2\}$,

$\overline{cl(y)} = \{x^j yz^k w^l; 0 \le j \le 2, 0 \le k, l \le 1\}$,

$\overline{cl(z)} = \{z^u w^v; 0 \leq u, v \leq 1, \quad u + v \neq 0\}$,

$\overline{cl(xz)} = \{x^i z^j w^k; 1 \leq i \leq 2, 0 \leq j, k \leq 1, \quad j + k \neq 0\}$.

Total number of fusion classes are 5
Fusion classes of $D_4 \times Z_3$

$D_4 \times Z_3 \cong < x, y, z: x^3 = y^4 = z^2 = 1, y^z = y^{-1}, xy = yx, xz = zx >$

Fusion classes are

$\overline{cl(1)} = \{1\}$,

$\overline{cl(x)} = \{x^i; 1 \leq i \leq 2\}$,

$\overline{cl(y)} = \{y^j; \quad j \in \{1,3\}\}$,

$\overline{cl(z)} = \{y^k z; 0 \leq k \leq 3\}$,

$\overline{cl(y^2)} = \{y^2\}$,

$\overline{cl(xy)} = \{x^i y^j; 1 \leq i \leq 2, \quad j \in \{1,3\}\}$,

$\overline{cl(xz)} = \{x^i y^j z; 1 \leq i \leq 2, 0 \leq j \leq 3\}$,

$\overline{cl(xy^2)} = \{x^i y^2; 1 \leq i \leq 2\}$.

Total number of fusion classes are 8
Fusion classes of $Q \times Z_3$

$Q \times Z_3 \cong < x, y, z: x^4 = z^3 = 1, x^2 = y^2, x^y = x^{-1}, yz = zy, xz = zx >$

Fusion classes are

$\overline{cl(1)} = \{1\}$,

$\overline{cl(x)} = \{x^i y^j; 0 \leq i \leq 3, 0 \leq j \leq 1\}$,

$\overline{cl(z)} = \{z^i; 1 \leq i \leq 2\}$,

$\overline{cl(xz)} = \{x^i y^j z^k; 0 \leq i \leq 3, 0 \leq j \leq 1, 1 \leq k \leq 2, (i,j) \neq (0,0), (2,0)\}$,

$\overline{cl(x^2)} = \{x^2\}$,

$\overline{cl(x^2 z)} = \{x^2 z^i; 1 \leq i \leq 1\}$.

So the total number of fusion classes are 6
Fusion classes of $A_4 \times Z_2$

$$A_4 \times Z_2 \cong < x, y, z, w: x^2 = y^2 = z^3 = w^2 = 1, xy = yx, zx = xyz, xw = wx, xz = zy, yw = wy, zw = wz >$$

Fusion classes are

$\overline{cl(1)} = \{1\}$,

$\overline{cl(x)} = \{x^i y^j; 0 \leq i,j \leq 1, \quad i+j \neq 0\}$,

$\overline{cl(z)} = \{x^l y^m z^n; 0 \leq l,m \leq 1, 1 \leq n \leq 2\}$,

$\overline{cl(w)} = \{w\}$,

$\overline{cl(xw)} = \{x^i y^j w; 0 \leq i,j \leq 1, \quad i+j \neq 0\}$,

$\overline{cl(zw)} = \{x^l y^m z^n w; 0 \leq l,m \leq 1, 1 \leq n \leq 2\}$.

So total number of fusion classes are 6
Fusion classes of Q_{12}

$Q_{12} =< a,b: a^{12} = 1, a^6 = b^2, b^{-1}ab = a^{-1} >$

Fusion classes are

$\overline{cl(1)} = \{1\}$,

$\overline{cl(a)} = \{a^i; 1 \leq i \leq 12, (i,12) = 1\}$,

$\overline{cl(a^2)} = \{a^i; \quad i \in \{2,10\}\}$,

$\overline{cl(a^3)} = \{a^i; \quad i \in \{3,9\}\}$,

$\overline{cl(a^4)} = \{a^i; \quad i \in \{4,8\}\}$,

$\overline{cl(a^6)} = \{a^6\}$,

$\overline{cl(b)} = \{a^j b; 0 \leq j \leq 11\}$.

So total number of fusion classes are 7
Fusion classes of D_{12}

$D_{12} =< a,b: a^{12} = b^2 = 1, bab = a^{-1} >$

Fusion classes are

$\overline{cl(1)} = \{1\}$,

$\overline{cl(a)} = \{a^i; 1 \leq i \leq 12, (i,12) = 1\}$,

$\overline{cl(a^2)} = \{a^i; \quad i \in \{2,10\}\}$,
$\overline{cl(a^3)} = \{a^i; \quad i \in \{3,9\}\}$,

$\overline{cl(a^4)} = \{a^i; \quad i \in \{4,8\}\}$,

$\overline{cl(a^6)} = \{a^6\}$,

$\overline{cl(b)} = \{a^j b; 0 \leq j \leq 11\}.$

So total number of fusion classes are 7
Fusion classes of $Z_6 \rtimes Z_4$

$Z_6 \rtimes Z_4 \cong < x, y: x^4 = y^6 = 1, yxy = x >$

Fusion classes are

$\overline{cl(1)} = \{1\},$

$\overline{cl(x)} = \{x^i y^j; \quad i \in \{1,3\}, 0 \leq j \leq 5\},$

$\overline{cl(y)} = \{x^k y^l; \quad k \in \{0,2\}, \quad l \in \{1,5\}\},$

$\overline{cl(x^2)} = \{x^2\},$

$\overline{cl(y^2)} = \{y^j; \quad j \in \{2,4\}\},$

$\overline{cl(y^3)} = \{x^i y^3; \quad i \in \{0,2\}\},$

$\overline{cl(x^2y^2)} = \{x^2 y^j; \quad j \in \{2,4\}\}.$

So total number of fusion classes are 7.

Fusion classes of $Z_3 \rtimes Z_8$

$Z_3 \rtimes Z_8 \cong < x, y: x^3 = y^8 = 1, yxy^{-1} = x^{-1} >$

Fusion classes are

$\overline{cl(1)} = \{1\},$

$\overline{cl(x)} = \{x^i; 1 \leq i \leq 2\},$

$\overline{cl(y)} = \{x^k y^l; 0 \leq k \leq 2, (2, l) = 1, 1 \leq l \leq 8\},$

$\overline{cl(y^2)} = \{y^j; \quad j \in \{2,6\}\},$

$\overline{cl(y^4)} = \{y^4\},$

$\overline{cl(xy^2)} = \{x^i y^j; 1 \leq i \leq 2, j \in \{2,6\}\},$

$\overline{cl(xy^4)} = \{x^i y^4; 1 \leq i \leq 2\}.$

So total number of fusion classes are 7
Fusion classes of $Z_3 \rtimes Q$

$Z_3 \rtimes Q \cong < x, y, z: x^2 = y^6 = z^2 = 1, xy = yx, xz = zx, (zy)^2 = 1 >$

Fusion classes are

$\overline{cl(1)} = \{1\}$,

$\overline{cl(x)} = \{x\}$,

$\overline{cl(y)} = \left\{x^l y^m; 0 \le l \le 1, \quad m \in \{1,5\}\right\}$,

$\overline{cl(z)} = \{x^i y^j z; 0 \le i \le 1, 0 \le j \le 5\}$,

$\overline{cl(y^2)} = \{y^j; \quad j \in \{2,4\}\}$,

$\overline{cl(y^3)} = \{x^i y^3; 0 \le i \le 1\}$,

$\overline{cl(xy^2)} = \left\{xy^j; \quad j \in \{2,4\}\right\}$

Thus, there are 7 fusion classes

Conclusion

Sr. No.	Group	order	$\lvert Aut(G)\rvert$	Number of fusion classes
				$k(G)$
	D_3	6	6	3
	D_4	8	8	4
	Q_3	8	24	3
	D_5	10	20	3
	$Z_3 \rtimes Z_4$	12	12	5
	A_4	12	24	3
	D_6	12	12	5
	D_7	14	42	3
	D_8	16	32	5
	G	16	16	6
	Q_{16}	16	32	5
	$D_4 \times Z_2$	16	64	5
	$Q_3 \times Z_2$	16	96	4
	$Modular - 16$	16	16	6
	B	16	32	6
	K	16	48	5
	$G_{4,4}$	16	32	5
	D_9	18	54	4
	$S_3 \times Z_3$	18	12	6
	$(Z_3 \times Z_3) \rtimes Z_2$	18	432	3
	D_{10}	20	40	5
	Fr_{20}	20	20	5
	$Z_5 \rtimes Z_4$	20	40	5
	Fr_{21}	21	42	4
	D_{11}	22	110	3
	$S_3 \times Z_4$	24	24	8
	$S_3 \times Z_2 \times Z_2$	24	144	5
	$D_4 \times Z_3$	24	48	6
	$Q \times Z_3$	24	48	6
	$A_4 \times Z_2$	24	24	6

	Q_{12}	24		
	D_{12}	24	48	7
	$Z_6 \rtimes Z_4$	24	48	7
	$Z_3 \rtimes Z_8$	24	24	7
	$Z_3 \rtimes Q$	24	48	7

Theorem The bounds of fusion classes of metabelian groups of order less than equal to 24 and given by $3 \leq k(G) \leq 8$.

Proof. Obvious from the observation obtained earlier.

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