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A review of arithmetic functions

Om Prakash Atale
Department of Mathematics, Khandesh College Education Society's Moolji Jaitha College, Jalgaon, Maharashtra, India

Abstract
In this paper, we have presented a brief review of arithmetic functions that appear in introductory number theory. First, we have discussed Modius function and Euler's totient function then we have provided a brief review on Mnagoldt function. Furthermore, theorems relating several arithmetic functions are also derived.

Keywords: arithmetic functions, mobius function, euler's totient function, mangoldt function, dirichlet series, riemann's zeta unction

Introduction
Arithmetic Functions
In number theory, an arithmetic function is just sequence of real or complex numbers defined on the set $\mathbb{N}$ of positive integers. Arithmetic functions play an important role in the study of divisibility properties of integers and the distribution of primes, since they are widely used in number theory, are also termed as number theoretic functions.

Definition
(Arithmetic functions)- *An arithmetic function is a function* $f:\mathbb{N}\to\mathbb{C}$ *defined on the set* $\mathbb{N}$ *of positive integers which takes values in real or complex numbers.*

Example
The Euler's totient function $\phi(n)$ counts the number of positive integers up to a given integer n that are relatively prime to n.

Example
The number of divisors function $\tau:\mathbb{Z}^+\to\mathbb{Z}^+$ counts the number of positive divisors of n and is denoted by $\tau(n)$.

Example
The sum of divisor function $\sigma:\mathbb{Z}^+\to\mathbb{Z}^+$ is the sum of all positive divisors of n and is denoted by $\sigma(n)$.

Mobius Function and Euler's Totient Function
The goal of this section is to introduce readers with two of the most important functions in number theory and the study of primes, that is, Mobius function $\mu(n)$ and Euler's totient function $\phi(n)$. We have stated, in this section, definition, values and few properties of both function along with their proofs along with some propositions in the end of this section.

Definition
(Mobius function)- *Mobius function is a number theoretic function* $\mu(n)$ *defined for all* $n\geq 1$ *and is defined as follows:*

$$\mu(1)=1.$$

If $n>1$, *then using the fundamental theorem of arithmetic which states that every integer* $n>1$ *can be represented as a product of prime factors in only one way apart from the order of factors, we define*

$$n=\prod_{i=1}^{k} {p_i}^{a_i}.$$

Then,

$$\mu(n) = (-1)^k$$

$$if\, a_1 = a_2 = \dots\dots = a_k = 1.$$
$$\mu(n) = 0$$

$$if\, n\ has\ a\ square\ factor > 1.$$

Mobius function have wide applications in study of distribution of primes and many other areas of number theory, we now state a remark on fundamental theorem of arithmetic and few theorems for multiplicity and inversion of mobius function.

Remark

There is a particularly rare drawback of the fundamental theorem of arithmetic in proving the Fermat's last theorem and that is, it holds for the number system that we use in our daily life or even in the number system that are most widely used in number theory, where any number can be factorized in essentially one way, but in the system of numbers that mathematicians wanted to use to try to solve Fermat's last problem, a different kind of number system was used where the fundamental theorem of arithmetic does not holds, numerous attempts were made by contemporary mathematicians but eventually it failed due to the unique factorization of numbers. German mathematician Ernst Kummer (1810-1893) studied the problem of the fundamental theorem of arithmetic and unique factorization extensively with the most beautiful results, but at the end, he was only able to solve up to 100 with exception the of 37, 59 and 67 where he use the method of infinite descent, but he was never able to solve it for a general class of number greater than 100, nor did any mathematician. The problem of Fermat's last theorem was eventually solved by English mathematician Andrew Wiles in 1995.

Table 1: Following is a short table of values of $\mu(n)$ ranging from 1 to 30

n	*μ (n)*	*n*	*μ* (n)	*n*	*μ* (n)
1	1	11	-1	21	1
2	-1	12	0	22	1
3	-1	13	-1	23	-1
4	0	14	1	24	0
5	-1	15	1	25	0
6	1	16	0	26	1
7	-1	17	-1	27	0
8	0	18	0	28	0
9	0	19	-1	29	-1
10	1	20	0	30	-1

Theorem

The function $\mu(n)$ *is a multiplicative function such that*

$$\mu(mn) = \mu(m)\mu(n)$$

for any pair of positive integer m *and* n *whenever* $gcd(m,n) = 1.$

Proof- Using the fundamental theorem of arithmetic, let

$$m = p_1p_2p_3 \dots\dots p_r$$

and

$$n = q_1q_2q_3 \dots\dots q_s.$$

Substituting the above values of m and n in $\mu(mn)$ and using definition 2.1, we get

$$\mu(mn) = \mu(p_1p_2p_3 \dots p_rq_1q_2q_3 \dots q_s)$$

$$= (-1)^{r+s}$$

$$= (-1)^r(-1)^s$$

$$= \mu(p_1p_2p_3 \dots p_r)\mu(q_1q_2q_3 \dots q_s)$$
$$= \mu(m)\mu(n).$$

Therefore, the function $\mu(n)$ is a multiplicative function, but not completely multiplicative, since $\varphi(4) = 0$ but $\varphi(2)\varphi(2) = 1$.

Theorem

For each positive integer $n \geq 1$,

$$\sum_{d|n} \mu(d) = \begin{cases} 1\,; if\ n = 1 \\ 0\,; if\ n > 1 \end{cases}$$

where d *runs through the positive divisors of* n.

Proof– consider the prime factorization of n and let

$$n = p_1^{a_1}p_2^{a_2}p_3^{a_3} \dots\dots p_r^{a_r}.$$

Now define

$$F(n) = \sum_{d|n} \mu(d) \qquad 1.1$$

such that if $\mu(d)$ is multiplicative, then $F(n)$ is also multiplicative. Therefore

$$F(n) = F(p_1^{a_1}p_2^{a_2}p_3^{a_3} \dots p_r^{a_r})$$

$$= F(p_1^{a_1})F(p_2^{a_2})F(p_3^{a_3}) \dots \dots F(p_r^{a_r})$$

From Eqn. (1.1), we know that

$$F(p_i^{a_i}) = \sum_{d|p_i^{a_i}} \mu(d),$$

where d is the divisor of $p_i^{a_i}$, therefore $d = 1, p_i, p_i^2, p_i^3 \dots \dots p_i^{a_i}$ and

$$F(p_i^{a_i}) = \mu(1) + \mu(p_i) + \mu(p_i^2) \dots \dots \mu(p_i^{a_i})$$

$$= 1 + (-1) + 0 + 0 \dots \dots + 0.$$

Therefore $F(p_i^{a_i}) = 0$ for $n > 1$, and hence

$$F(n) = \sum_{d|n} \mu(d) = 0$$

for $n > 1$. In the case of $n = 1$ the only possibility is that all the powers of the primes are equal to zero, in that case

$$n = p_1^{a_1}p_2^{a_2}p_3^{a_3} \dots\dots p_r^{a_r} = 1$$

if

$$a_1 = a_2 = a_3 \dots \dots = a_r = 0,$$

and hence

$$F(1) = \sum_{d|1} \mu(d) = \mu(1)$$

$$F(1) = 1.$$

Theorem
can also be written in form

$$\sum_{d|n} \mu(d) = \left[\frac{1}{n}\right] = \begin{cases} 1\,; n = 1 \\ 0\,; n > 1 \end{cases} \quad 1.2$$

where $[x]$ is the greatest integer function which denotes the greatest integer $\leq x$.

Theorem
(Mobius inversion formula)- *Let f and F be two number theoretic functions related by the formula*

$$F(n) = \sum_{d|n} f(d) \quad 1.3$$

for every positive integer n, then

$$f(n) = \sum_{d|n} \mu(d)\, F\left(\frac{n}{d}\right) = \sum_{d|n} \mu\left(\frac{n}{d}\right) F(d).$$

Proof- Considering that d is a divisor of n, we can write $n = dd'$, which can also be written as $d' = n/d$, which means that d' is also a divisor of n. Using Eqn. (1.3) and replacing the notation for divisor d to c for the sake of avoiding confusion, we have

$$f(d) = \sum_{d|n} \mu(d)\, F\left(\frac{n}{d}\right) = \sum_{d|n} \mu(d) \sum_{c|\left(\frac{n}{d}\right)} f(c)$$

$$= \sum_{d|n} \sum_{c|\left(\frac{n}{d}\right)} \mu(d) f(c)$$

Now, $d|n$ and $c|\left(\frac{n}{d}\right)$ if $c|n$ and $d|\left(\frac{n}{c}\right)$, therefore

$$\sum_{d|n} \sum_{c|\left(\frac{n}{d}\right)} \mu(d) f(c) = \sum_{c|n} f(c) \sum_{d|\left(\frac{n}{c}\right)} \mu(d)$$

$$= \sum_{\substack{c|n \\ c<n}} f(c) \sum_{\substack{d|\left(\frac{n}{c}\right) \\ c<n}} \mu(d) + \sum_{\substack{c|n \\ c=n}} f(c) \sum_{\substack{d|\left(\frac{n}{c}\right) \\ c=n}} \mu(d).$$

Using theorem 2.2, we obtain

$$\sum_{\substack{d|\left(\frac{n}{c}\right) \\ c<n}} \mu(d) = 0$$

and

$$\sum_{\substack{d|\left(\frac{n}{c}\right) \\ c=n}} \mu(d) = 1,$$

Therefore

$$\sum_{\substack{c|n \\ c<n}} f(c) \sum_{\substack{d|\left(\frac{n}{c}\right) \\ c<n}} \mu(d) + \sum_{\substack{c|n \\ c=n}} f(c) \sum_{\substack{d|\left(\frac{n}{c}\right) \\ c=n}} \mu(d) = 0 + f(n).$$

The other equality in theorem 2.3 can be obtained by similar method.
The function was introduced by Möbius (1832), and the notation $\mu(n)$ was first used by Mertens (1874). One if its most important applications is in the formulas for the Euler totient, which is the next function I will define, but before stating the definition of Euler's totient function, we need to understand the definition of *"two numbers*

being relatively prime to each other". We say that two numbers n and m are relatively prime to each other when $\gcd(n, m) = 1$, where $\gcd$ is the greatest common divisor which divides both n and m.

Definition

(Euler's totient function)- *The Euler's totient function* $\phi(n)$ *is defined to be the number of positive integers which are less or equal to an integer and are relatively prime to any Integer* n. *For* $n \geq 1$, *the Euler's totient function is*

$$\phi(n) = \sum_{k=1}^{n}{}' 1$$

Where the ' indicates that the sum is only over the integers which are relatively prime to n.

Table 2: Following is a short table of values of $\phi(n)$ ranging from 1 to 30:

n	$\phi(n)$	*n*	$\phi(n)$	*n*	$\phi(n)$
1	1	11	10	21	12
2	1	12	4	22	10
3	2	13	12	23	22
4	2	14	6	24	8
5	4	15	8	25	20
6	2	16	8	26	12
7	6	17	16	27	18
8	4	18	6	28	12
9	6	19	18	29	28
10	4	20	0	30	8

Now, we state 3 theorems which will later be used in deriving certain definitions related to rigorous calculations concerned with Euler's totient function.

Theorem

For $n \geq 1$, *we have*

$$\sum_{d|n} \phi(d) = n.$$

The above equation is what we know as the divisor formula which is most important in the study of Euler's totient function, in short, what it states that the sum of all the totient functions for the divisors of n, is equal to n, where $n \geq 1$.

Theorem

For $n \geq 1$, *we have*

$$\phi(n) = \sum_{d|n} \mu(d) \frac{n}{d}.$$

Proof- There are two ways of deriving theorem 2.5, first is by using the Mobius inversion formula and second proof involves Eqn. (1.2)

By Mobius inversion formula we know that for every positive integer n,

$$f(n) = \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right).$$

Define $F(n) = n$ and $f(n) = \phi(n)$, therefore, we get

$$\phi(n) = \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right) = \sum_{d|n} \mu(d) \frac{n}{d}.$$

We start our second proof with definition 2.2 which can also be written in the form

$$\phi(n) = \sum_{k=1}^{n} \left[\frac{1}{(n,k)}\right],$$

where $[x]$ is the greatest integer function which denotes the greatest integer $\leq x$ and (n,k) is same as $\gcd(n,k)$ which denotes the greatest common divisor of n and k. Now replacing n with (n,k) in Eqn. (1.2), we get

$$\phi(n) = \sum_{k=1}^{n} \sum_{d|(n,k)} \mu(d) = \sum_{k=1}^{n} \sum_{\substack{d|n \\ d|k}} \mu(d),$$

For a fixed divisor d of n we must sum over all those k in the range $1 \leq k \leq n$ Which are multiples of d. Therefore, we can write $k = qd$ then the inequality $1 \leq k \leq n$ holds if and only if $1 \leq q \leq n/d$. Therefore

$$\sum_{k=1}^{n} \sum_{\substack{d|n \\ d|k}} \mu(d) = \sum_{d|n} \sum_{q=1}^{\frac{n}{d}} \mu(d) = \sum_{d|n} \mu(d) \sum_{q=1}^{n/d} 1 = \sum_{d|n} \mu(d) \frac{n}{d}.$$

The second proof of theorem 2.5 can also be found in “Introduction to Analytic Number Theory” by Tom M. Apostol [1. Pg. 26].

Theorem
For $n \geq 1$, *we have*

$$\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right).$$

Proof- Since the Euler’s totient function is the number of positive integers that are relatively prime to n, we can write $\phi(n)$ as a product of prime divisors of n. Where prime divisors of n are those integers which are divisible by prime numbers.
In the case of $n = 1$, we know that there is no prime number that divides 1, therefore, the product remains empty in the case where $n = 1$.
Now, consider the case where $n > 1$. we know from the fundamental theorem of arithmetic that any integer n can be written as a product of distinct prime divisors of n, therefore, we can write

$$n = p_1 p_2 p_3 \ldots\ldots p_r.$$

The product can now be written as

$$\prod_{p|n} \left(1 - \frac{1}{p}\right) = \prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right).$$

The second term in the product of righthand side which contain prime divisor p_i in its denominator, when gets multiplied with other terms after expanding the product, contains the product of distinct prime divisors of n in the denominator of almost every term in the obtained product. Therefore

$$\prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right) = 1 - \sum \frac{1}{p_i} + \sum \frac{1}{p_i p_j} - \sum \frac{1}{p_i p_j p_k} + \cdots + \frac{(-1)^r}{p_1 p_2 p_3 \ldots\ldots p_r}, \quad 1.4$$

where the terms like $p_i p_j p_k$ contains all the possible products of distinct prime divisors of n. Every term in the denominator of the righthand side of the Eqn. (1.4) contains the prime divisor of n in the form of multiplication of distinct primes, therefore, each term can be written in the form $\pm 1/d$, where d is the divisor of n which is either 1 or the product of distinct primes and using definition 2.1, we can observe that the numerator ± 1 is exactly $\mu(d)$. Since $\mu(d)$=0 when d has a square factor > 1, the sum in Eqn. (1.4) can be written in the form

$$\sum_{d|n} \frac{\mu(d)}{d}.$$

Therefore,

$$n\prod_{p|n}\left(1-\frac{1}{p}\right)=\sum_{d|n}\mu(d)\frac{n}{d}.$$

From theorem 2.5, we know that for $n \geq 1$, we have

$$\phi(n)=\sum_{d|n}\mu(d)\frac{n}{d}.$$

Therefore,

$$\phi(n)=n\prod_{p|n}\left(1-\frac{1}{p}\right),$$

which completes the proof.

Theorem
Euler's totient function has the following properties

(a) $\phi(p^{\alpha})=p^{\alpha}-p^{\alpha-1}$ for prime p and $\alpha \geq 1$.
(b) $\phi(mn)=\phi(m)\phi(n)(d/\phi(d))$ where $d=\gcd(m,n)$.
(c) $\phi(mn)=\phi(m)\phi(n)$ if $\gcd(m,n)=1$.
(d) $a|b$ implies $\phi(a)|\phi(b)$.
(e) $\phi(n)$ is even for $n \geq 3$. Moreover, if n has r distinct odd prime factors, then $2^r|\phi(n)$.
Proof of theorem 2.7 can be found in [1. Pg. 28].

Proposition
For a positive integer n, we have

$$\frac{n}{\phi(n)}=\sum_{d|n}\frac{\mu^2(d)}{\phi(d)},$$

where the sum is over all the divisors of n.

Proposition
For all n with at most 8 distinct prime factors, we have

$$\phi(n)>\frac{n}{6}.$$

Proposition
Let $f(x)$ be defined for all rational x in $0 \leq x \leq 1$ and let

$$F(n)=\sum_{k=1}^{n} f\left(\frac{k}{n}\right)$$

$$F^{*}(n)=\sum_{\substack{k=1\\(k,n)=1}}^{n} f\left(\frac{k}{n}\right),$$

then

A. $F^{*}=\mu * F$, the Dirichlet product of μ and F.
B. $\mu(n)$ is the sum of the primitive n^{th} root of unity:

$$\mu(n)=\sum_{\substack{k=1\\(k,n)=1}}^{n} e^{2\pi ik/n}.$$

A detailed proof of proposition 2.1, 2.2 and 2.3 can be found in [2].

Mangoldt Function

The von Mangoldt function or simply Mangoldt function [3], denoted by $\Lambda(n)$, is an arithmetic function that plays a critical role in the distribution of primes.

Definition

(Mangoldt function)- *For every integer* $n \geq 1$,*we define*

$$\Lambda(n) = \begin{cases} \log p & ; if\ n = p^m, p\ prime, m \geq 1 \\ 0 & ; otherwise. \end{cases}$$

Table 3: Following is a short table of values of $\Lambda(n)$ ranging from 1 to 10:

n	$\Lambda(n)$
1	0
2	$\log 2$
3	$\log 3$
4	$\log 2$
5	$\log 5$
6	0
7	$\log 7$
8	$\log 2$
9	$\log 3$
10	0

Theorem

For $n \geq 1$, we have

$$\log n = \sum_{d|n} \Lambda(d)\,. \tag{1.5}$$

Proof- Since $\log 1 = 0$, in the case were $n = 1$ both sides of the equations are zero. For $n > 1$, using the fundamental theorem of arithmetic, we can write n as a product of primes, therefore

$$n = \prod_{k=1}^{r} {p_k}^{a_k}.$$

Taking logarithm both the sides, we get

$$\log n = \sum_{k=1}^{r} a_k \log p_k.$$

The nonzero terms in the sum of Eqn. (1.8) comes from those divisors which obeys the fundamental theorem of arithmetic, that is, the divisors of the form ${p_k}^m$ for $m = 1,2,3 \ldots a_k$ and $k = 1,2,3 \ldots r$. Therefore, we can expand the sum in Eqn. (1.8) as

$$\sum_{d|n} \Lambda(d) = \sum_{k=1}^{r} \sum_{m=1}^{a_k} \Lambda({p_k}^m) = \sum_{k=1}^{r} \sum_{m=1}^{a_k} \log p_k = \sum_{k=1}^{r} a_k \log p_k = \log n,$$

which completes the proof.

Theorem

For $n \geq 1$, *we have*

$$\Lambda(n) = \sum_{d|n} \mu(d) \log \frac{n}{d} = -\sum_{d|n} \mu(d) \log d.$$

Proof- Applying Mobius inversion formula on Eqn. (1.5) with $f(n) = \Lambda(n)$ and $F(n) = \log n$, we get

$$\Lambda(n) = \sum_{d|n} \mu(d) \log \frac{n}{d} = \log n \sum_{d|n} \mu(d) - \sum_{d|n} \mu(d) \log d,$$

Since $\log n \sum_{d|n} \mu(d) = 0$ for all n, we get

$$\Lambda(n) = \sum_{d|n} \mu(d) \log \frac{n}{d} = -\sum_{d|n} \mu(d) \log d,$$

which completes our proof.

Remark

The Mangoldt function is closely connected with the Riemann zeta function $\zeta(s)$. In fact, the generating series for $\Lambda(n)$ is the logarithmic derivative of $\zeta(s)$:

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n} \Lambda(n) n^{-s} \quad (\Re s > 1)$$

The Mangoldt function was proposed by H. Mangoldt in 1894.

Dirichlet Series and Riemann Zeta Function

The goal of this short section is to show a relation between Dirichlet series [4] and the Riemann zeta function such that Riemann zeta function is a particular case of Dirichlet series. Furthermore, we also stated a remark on the convergence and zeroes of Riemann zeta function. Let s be a complex variable such that $s = \sigma + it$ for all $\sigma, t \in \mathbb{R}$. We shall always use σ for $\Re(s)$ and t for $\Im(s)$.

Definition

(Dirichlet series)- *If* $f: \mathbb{N} \to \mathbb{C}$ *has at most a polynomial growth, then for* $s \in \mathbb{C}$, *Dirichlet series associated with* f *can be written as*

$$D_f(s) = \sum_{n=1}^{\infty} f(n) n^{-s}.$$

Remark

The product of Dirichlet series is a Dirichlet series, therefore

$$D_f(s) D_g(s) = D_h(s).$$

Definition

The constant function $f(n) = 1$ *for all* $n \in \mathbb{N}$ *has the Dirichlet series of the form*

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s},$$

which is called the Riemann zeta function.

Remark

The Dirichlet series for $\zeta(s)$ converges if $\sigma > 1$, in fact it converges absolutely for such s, since

$$|n^{-s}| = \left|e^{-(\sigma+it)\log n}\right| = n^{-\sigma}.$$

Also, if $\sigma \leq 0$ or $0 < s \leq 1$, the series diverges, in the first case because the terms do not tend to zero, in the second by comparison with the harmonic series.

Remark

If Ω_p denotes the open half plane such that $\Omega_p = \{s : \Re(s) > p\}$, then $\zeta(s)$ has no zeroes in Ω_1.

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