International Journal of Physics and Mathematics
www.physicsjournal.net
Online ISSN: 2664-8644, Print ISSN: 2664-8636
Received: 02-03-2022, Accepted: 17-03-2022, Published: 02-04-2022
Volume 4, Issue 1, 2022, Page No. 18-20

Some basic properties of completely monotonic functions

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Abstract
In this article, we review some basic results on the class of completely monotonic functions. We also introduce the relationship among absolutely monotonic functions, completely monotonic sequences, and completely monotonic functions; and the compositions of completely monotonic functions and absolutely monotonic functions.

Keywords: completely monotonic functions, completely monotonic sequences

Introduction

Let's first introduce the notion of absolutely monotonic functions, which is closely related to that of completely monotonic functions.
Bernstein [1] in 1914 first introduced

Definition A. A function f is said to be absolutely monotonic on an interval I, if $f \in C(I)$, has derivatives of all orders on I^o and for all $n \in \mathbb{N}_0$

$$f^{(n)}(x) \geq 0, \quad x \in I^o.$$

Here, and throughout the paper, $C(I)$ is the set of all continuous functions on the interval I, I^o is the interior of the interval I, $\mathbb{N}$ is the set of all positive integers, $\mathbb{N}_0$ is the set of all non-negative integers, $\mathbb{R}^+$ is the set of all positive real numbers.
We use $AM(I)$ to denote the class of all absolutely monotonic functions on the interval I.

Definition B ([10])**.** A function f is said to be completely monotonic on an interval I, if $f \in C(I)$, has derivatives of all orders on I^o and for all $n \in \mathbb{N}_0$

$$(-1)^n f^{(n)}(x) \geq 0, \quad x \in I^o. \tag{1}$$

We use $CM(I)$ to denote the class of all completely monotonic functions on the interval I.
From Definitions A and B, we have

Theorem C ([10]). A function f is completely monotonic on an interval I if and only if the function $f(-x)$ is absolutely monotonic on the interval $-I$.
If $f, g \in CM(I)$, and $\alpha, \beta > 0$, then by definition $\alpha f + \beta g \in CM(I)$, and by Leibniz's rule $fg \in CM(I)$.
For the operation of point-wise convergence, we have (see [10, p. 151])

Theorem D ([10]). Suppose that for $n \in \mathbb{N}$, $f_n \in CM(I)$, where $I := (a, b)$ or $(a, b]$. If the limit function $f(x) := \lim_{n\to\infty} f_n(x)$ exists on I, then $f \in CM(I)$.
Clearly this theorem is equivalent to

Theorem E ([10]). Suppose that for $n \in \mathbb{N}$, $f_n \in CM(I)$, where $I := (a, b)$ or $(a, b]$. If the function $f(x) := \sum_{n=1}^{\infty} f_n(x)$ exists on I, then $f \in CM(I)$.

The following two results can also be found in [10]:
Theorem F ([10]). If $f(x)$ is completely monotone on $(a, b]$, then it can be extended analytically into the region of the complex z-plane: $|z - b| < b - a$, where $z = x + iy$.

Theorem G ([10]). Suppose that $f \in CM(I)$. If there exists $x_0 \in I^o$ such that $f(x_0) = 0$, then $f(x) \equiv 0$ on I.
Dubourdieu [3] in 1939 showed

Theorem H ([3]). A non-constant completely monotonic function on (a, ∞) is strictly completely monotonic there.
In 1963 Lorch and Szego [7] gave an alternative proof of this result.
In 1987 O'Cinneide [9] proved

Theorem I ([9]). If $f \in CM(\mathbb{R}^+)$ then $f(x+\delta)/f(x)$ is strictly increasing in x on $\mathbb{R}^+$ for each $\delta > 0$ unless $f(x) = ce^{-dx}$ for some $c \geq 0$ and some $d \geq 0$.
In 1939 Feller [4] showed

Theorem J ([4]). Suppose that $f, g \in CM(\mathbb{R}^+)$. If there exists a strictly increasing sequence $\{x_k\}_1^\infty \subset \mathbb{R}^+$ with $\sum_{k=1}^\infty (1/x_k)$ diverging such that $f(x_k) = g(x_k), k \in \mathbb{N}$, then $f \equiv g$ on $\mathbb{R}^+$.
The following result is Theorem 3 of [11].

Theorem K ([11]). If $f \in CM(I)$, where $I := (a, \infty)$, then for all $n \in \mathbb{N}_0$

$$[f(x), f'(x), \cdots, f^{(n)}(x)] \geq 0, \quad x \in I. \tag{2}$$

Here, in (2),

$$[f(x), f'(x), \cdots, f^{(n)}(x)] := \begin{vmatrix} f(x) & f'(x) & \cdots & f^{(n)}(x) \\ f'(x) & f''(x) & \cdots & f^{(n+1)}(x) \\ \vdots & \vdots & \ddots & \vdots \\ f^{(n)}(x) & f^{(n+1)}(x) & \cdots & f^{(2n)}(x) \end{vmatrix}$$

Is a Hankel determinant (see [10]).
From this result, we can show

Theorem L. Suppose that $f \neq 0$ on $I := (a, \infty)$ and that $f \in CM(I)$, then f is log-convex there.
There exists a close relationship between completely monotonic functions and completely monotonic sequences.
In 1931 Widder [11] showed

Theorem M ([11]). Suppose that $f \in CM[a, \infty)$, then for any $\delta \geq 0$, the sequence $\{f(a + n\delta)\}_{n=0}^\infty$ is completely monotonic.
This result was generalized by Lorch and Newman [6] in 1983 as follows

Theorem N ([6]). Suppose that $f \in CM[a, \infty)$. If the sequence $\{\Delta x_k\}_{k=0}^\infty$ is completely monotonic and $x_0 \geq a$, then so is the sequence $\{f(x_k)\}_{k=0}^\infty$.
They [6] also proved the following two results.

Theorem O ([6]). If $f \in AM[0, \infty)$ and if the sequence $\{x_k\}_{k=0}^\infty$ is completely monotonic, then so is the Sequence $\{f(x_k)\}_{k=0}^\infty$.

Theorem P ([6]). Suppose that $f' \in CM(\mathbb{R}^+)$ and that the sequence $\{\Delta x_k\}_{k=0}^\infty$ is completely monotonic. If x_0 is in the domain of f, then the sequence $\{\Delta f(x_k)\}_{k=0}^\infty$ is completely monotonic.
Now suppose that $f \in CM[0, \infty)$, by Theorem M $\{f(n)\}_0^\infty$ is completely monotonic. We may ask if there exists an interpolating function $f \in CM[0, \infty)$ such that $f(n) = \mu_n, n \in \mathbb{N}_0$ for any given completely monotonic sequence $\{\mu_n\}_0^\infty$. For this, Widder [11] in 1931 established

Theorem Q ([11]). There exists a function $f \in CM[0, \infty)$ such that $f(n) = \mu_n, n \in \mathbb{N}_0$ if and only if the sequence $\{\mu_n\}_0^\infty$ is minimal completely monotonic.
For compositions of completely monotonic and related functions, the following two results are a version of the corresponding Theorems in [10, Chapter IV].

Theorem S ([10]). Suppose that $f \in AM(I_1), g \in AM(I)$ and $\mathcal{R}(g) \subset I_1$, then $f \circ g \in AM(I)$.

Theorem T ([10]). Suppose that $f \in AM(I_1), g \in CM(I)$ and $\mathcal{R}(g) \subset I_1$, then $f \circ g \in CM(I)$.
The next result, which was established in 1983 by Lorch and Newman
[6, Theorem 5], is a converse of Theorem T.

Theorem U ([6]). If for each $g \in CM(\mathbb{R}^+)$, $f \circ g \in CM(\mathbb{R}^+)$, then $f \in AM(\mathbb{R}^+)$.
The following result is a generalized form of Theorem 2 of [8].

Theorem V. Suppose that $f \in CM(I_1), g \in C(I), g' \in CM(I^o)$ and $\mathcal{R}(g) \subset I_1$, then $f \circ g \in CM(I)$.
In 1983 Lorch and Newman [6, Theorem 4] gave an interesting result related to Theorem V as follows.

Theorem W ([6]). For each function $f \in CM(I)$, where $I := [0, \infty)$, there exists a function g on I such that $g(0) = 0, f \circ g \in CM(I)$ and $g' \notin CM(\mathbb{R}^+)$.

For the representations of the completely monotonic functions on $\mathbb{R}^+$ or on $[0, \infty)$, the following are the well Known Bernstein's Theorems (see [10, Chapter IV]).

Theorem X ([10]). $f \in CM(\mathbb{R}^+)$ if and only if there exists an increasing function $\alpha(t)$ on $[0, \infty)$ such that

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t). \tag{3}$$

Theorem Y ([10]). $f \in CM(I)$, Where $I := [0, \infty)$, if and only if there exists a bounded, increasing function $\alpha(t)$ on I such that

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t).$$

Here, and throughout the thesis, increasing and decreasing are understood in a non-strict sense, i.e. increasing means non-decreasing (and decreasing means non-increasing). The right side of (3) is an improper Stieltjes integral.

In [10] Widder provided three different methods of proof for Theorem Y.
In 1928, Bernetein [2] proved the following result.

Theorem Z ([2]). $f \in AM(I)$, where $I := (-\infty, 0)$, if and only if there exists an increasing function $\alpha(t)$ on $[0, \infty)$ such that

$$f(x) = \int_0^\infty e^{xt} d\alpha(t).$$

Also Hausdorff [5] in 1921 established a similar result to this theorem.

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