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Spherically symmetric model with electromagnetic field in stationary space-time

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Abstract
We have evaluated spherically symmetric model with electromagnetic field and with equation of state $p = \gamma\rho$, where γ is arbitrary constant, by solving Einstein's field equations in stationary space-time. It is observed that dust, radiating and stiff dominated spherically symmetric models does not exist. Spherically symmetric dark energy models exist and they are studied in regards with their geometrical and physical aspects in detail.

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Introduction

Symmetry has played a very significant role in the formulation of physical theories and in the various relativistic theories of gravitation. The spherical symmetry has its own importance in general theory of relativity due to its comparative simplicity. A universe with a spherical geometry has a finite size but no boundary. It has positive curvature and all points in it are equivalent. By having spherical symmetry, we can simplify the equations of motions considerably. Spherically symmetric space-time is described as one whose metric is invariant under Rotations. Many important space-times like Schwarzschild solutions (exterior and interior), the Robertson – Walker model of the expanding universe etc. are all spherically symmetric. Many researchers [1 – 10] have studied and discussed the nature of various forms of spherically symmetric space-times in relativistic gravitational theories with different parameters and conditions.

The occurrence of magnetic field on galactic scale is a well-established fact today and magnetic field models have significant contribution in the study of various astrophysical phenomenons and evolution of galaxies. Harrison [11] has suggested that magnetic field could have a cosmological origin. Melvin [12] has described that the presence of magnetic fields is not as unrealistic as it appears to be, because during the evolution of universe, the matter is in highly ionized state and it is smoothly coupled with the field to form neutral matter due to universe expansion. Upadhaya *et al.* [13], Pradhan *et al.* [14 – 15], Tyagi *et al.* [16], Singh *et al.* [17], Bali *et al.* [18], Bhoyar *et al.* [19], Charjan *et al.* [20] and Banerjee *et al.* [21] are some researchers who have studied the effect of magnetic field in the evolution of the universe.

In the universe, 96 % of energy content is in exotic form, out of which 70 % of energy is repulsive in nature, called dark energy and 23 % of energy is attractive in nature, called dark matter [22 – 23]. Although dark matter is the most popular theory to explain the various astronomical observations of galaxies and galaxy clusters, there has been no direct observational evidence of dark matter. Dark matter is matter that is inferred to exist from gravitational effects on visible matter and background radiation, but is undetectable by emitted or scattered electromagnetic radiation [24] whereas dark energy is a hypothetical form of energy that permeates all of space and tends to increase the rate of expansion of universe [25]. Generally, dark energy has a strong negative pressure (*i.e.* effects, acting repulsively) in order to explain the observed acceleration in the expansion rate of the universe. Dark energy is one of the popular ways to explain recent observations and experiments that the universe appears to be expanding at an accelerating rate. Recently, the models of dark energy have created a lot of interest in the research area and many researchers [26 – 33] have developed the dark energy cosmological models of the universe.

Stationary space-times or Time – independent gravitational fields are very significant in general relativity. In a stationary space-time, the metric tensor components g_{ij}, are all independent of the time coordinate. The geometry of a stationary space-time does not change in time and one can find a family of observers who observe no change in the gravitational field (or sources such as matter or electromagnetic fields) over time. It corresponds to the gravitational field generated by a time – independent source with rotation, while a static space-time is generated by a time – independent source without rotation. Rotating black holes are stationary and non – rotating black holes are static.

Many researchers are studying the nature and properties of the different cosmological models of the universe in stationary space-times. Beig *et al.* [34] have thrown light on the far – field behavior of stationary space-times. Garcia *et al.* [35] have discussed the conformally flat stationary axisymmetric space-times. Wu *et al.* [36] have studied the Newman – Penrose constants of stationary space-times. Katz *et al*. [37] have obtained expression for gravitational energy in stationary space-times. Javaloyes *et al*. [38] have investigated the existence of standard splittings for conformally stationary space-times. Many other authors like Chrusciel *et al*. [39], Nayak [40] and Borkar *et al*. [41 – 43] have obtained models of universe in stationary space-time with different parameters and conditions.

Dhongle [44] has studied the dark energy of de-Sitter universe and has obtained that the de-Sitter universe is completely filled with dark energy and dark matter and it is the generalization of empty de-Sitter universe. We extend this work with arbitrary equation of state parameter γ and with the assumption of electromagnetic field. In this paper, we obtain the spherically symmetric model with electromagnetic field and with equation of state $p = \gamma\rho$, where γ is arbitrary constant, by solving Einstein's field equations in stationary space-time. It is observed that dust, radiating and stiff dominated spherically symmetric models does not exist. Spherically symmetric dark energy models exist and their geometrical and physical aspects are studied in detail.

The Metric and The Einstein's Field Equations

We consider the spherically symmetric metric in polar form

$$d\tau^2 = -e^{\lambda} dr^2 - r^2 d\theta^2 - r^2 Sin^2\theta\, d\phi^2 + e^{\nu} dt^2 , \quad (1)$$

Where ν and λ are functions of r and t. The Einstein's field equations are

$$R_{ij} - \frac{1}{2} R\, g_{ij} = 8\pi\, T_{ij} \quad (2)$$

Assume that the space-time is field with matter consisting of perfect fluid with electromagnetic field, whose energy-momentum tensor T_{ij} is given by

$$T_{ij} = (\rho + p)\, u_i u_j - p\, g_{ij} + E_{ij} , \quad (3)$$

In which the electromagnetic tensor E_{ij} is

$$E_{ij} = \frac{1}{4\pi}\left[-F_{li} F_{mj}\, g^{lm} + \frac{1}{4} F_{lm} F^{lm} g_{ij} \right]. \quad (4)$$

Here ρ is the energy density and p is the pressure of the perfect fluid. We choose the components of four-velocity u^i as $u^1 = 0 = u^2 = u^3$ and $u^4 \neq 0$. Here, we have

$$g_{ij}\, u^i u^j = 1 \quad (5)$$

The electromagnetic field tensor F^{ij} satisfies

$$F^{ij}_{\ \ |j} = 4\pi \in u^i \quad (6)$$

and Maxwell's equations

$$F_{[ij\,|\,k]} = 0 \quad (7)$$

Here stroke '|' denotes covariant differentiation and $\in$ is the current density. We assume that current is flowing along x – axis, so that F^{14} is the only non-vanishing component of the electromagnetic field tensor F^{ij}.

The Einstein's field equations (2) in stationary space-time take the form

$$P_{\alpha\beta} + \frac{h}{2} f_{\alpha}^{\ \gamma} f_{\beta\gamma} - \frac{1}{2}\gamma_{\alpha\beta} P = 8\pi\, T_{\alpha\beta} , \quad (8)$$

$$\frac{3}{8} h f_{\alpha\beta} f^{\alpha\beta} + \frac{1}{2} P = \frac{8\pi}{h} T_{44}, \tag{9}$$

And

$$\frac{\sqrt{h}}{2} f_{\alpha \ |\beta}^{\ \beta} + \frac{3}{2} f_{\alpha\beta} (\sqrt{h})^{|\beta} = \frac{8\pi}{\sqrt{h}} T_{4\alpha}, \tag{10}$$

Where

$$\gamma_{\alpha\beta} = -g_{\alpha\beta} + \frac{g_{4\alpha}\, g_{4\beta}}{h}, \quad \alpha, \beta = 1, 2, 3 \tag{11}$$

is the three-dimensional metric tensor determining the geometry of space, $f_{\alpha\beta}$ is the three – dimensional anti-symmetric tensor given by

$$f_{\alpha\beta} = g_{\beta\,|\alpha} - g_{\alpha\,|\beta} = \frac{\partial g_\beta}{\partial x^\alpha} - \frac{\partial g_\alpha}{\partial x^\beta} \tag{12}$$

And

$$h = g_{44}, \quad g_\alpha = \frac{-g_{4\alpha}}{h}, \quad \alpha = 1, 2, 3. \tag{13}$$

P is the three - dimensional scalar curvature given by

$$P = \gamma^{\alpha\beta} P_{\alpha\beta} \tag{14}$$

Where $P_{\alpha\beta}$ is the three - dimensional Ricci tensor constructed from the three – dimensional metric tensor $\gamma_{\alpha\beta}$ in the same way as R_{ik} is constructed from the g_{ik} [45].

In stationary space-times, the gravitational potentials g_{ij} are independent of time *t*. Hence, for the metric (1) in stationary space-time, λ and ν are functions of *r* alone. From equation (5), we get $u^4 = e^{-\nu/2}$. Thus the Einstein's field equations (8 – 10) in stationary space-times becomes

$$\frac{1}{r^2} e^\lambda - \frac{1}{r^2} = -8\pi\, p e^\lambda + e^{-\nu} F_{14}^2, \tag{15}$$

$$\frac{1}{2} \frac{\lambda'}{r} = -8\pi\, p e^\lambda \,\square\, e^{-\nu} F_{14}^2, \tag{16}$$

$$\frac{1}{r^2} e^\lambda - \frac{1}{r^2} + \frac{\lambda'}{r} = 8\pi\, \rho\, e^\lambda + e^{-\nu} F_{14}^2 \tag{17}$$

3. Solutions of Field Equations and Significance

There are three differential equations (15), (16) and (17) in five unknowns λ, ν, p, ρ and F_{14}. To have a unique solution, we take two extra conditions. First we assume the equation of state as

$$p = \gamma\, \rho \tag{18}$$

where γ is arbitrary constant and second we assume the relation between the metric potential as

$$\nu = -\lambda \tag{19}$$

In view of these two conditions (18) and (19), the differential equations (15), (16) and (17) take the form

$$\frac{1}{r^2}e^{\lambda} - \frac{1}{r^2} = -8\pi\gamma\,\rho\, e^{\lambda} + e^{\lambda}F_{14}^{2}, \tag{20}$$

$$\frac{1}{2}\frac{\lambda'}{r} = -8\pi\gamma\,\rho\, e^{\lambda} - e^{\lambda}F_{14}^{2}, \tag{21}$$

$$\frac{1}{r^2}e^{\lambda} - \frac{1}{r^2} + \frac{\lambda'}{r} = 8\pi\,\rho\, e^{\lambda} + e^{\lambda}F_{14}^{2} \tag{22}$$

Adding equations (21) and (22) gives

$$\frac{1}{1-\gamma}\left\{\frac{1}{r^2}e^{\lambda} - \frac{1}{r^2} + \frac{3}{2}\frac{\lambda'}{r}\right\} = 8\pi\,\rho\, e^{\lambda} \tag{23}$$

and from equations (20) and (21), we write

$$\frac{1}{\gamma}\left\{\frac{1}{2r^2}e^{\lambda} - \frac{1}{2r^2} + \frac{1}{4}\frac{\lambda'}{r}\right\} = -8\pi\,\rho\, e^{\lambda} \tag{24}$$

The equations (23) and (24) after simplification gives

$$e^{\lambda} = \left(1 - A\,r^{\frac{-2(1+\gamma)}{1+5\gamma}}\right)^{-1} \tag{25}$$

and

$$e^{\nu} = \left(1 - A\,r^{\frac{-2(1+\gamma)}{1+5\gamma}}\right), \tag{26}$$

Where $A(\geq 0)$ is the constant of integration. Thus the required metric is

$$d\tau^2 = -\left(1 - A\,r^{\frac{-2(1+\gamma)}{1+5\gamma}}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 Sin^2\theta\, d\phi^2 + \left(1 - A\,r^{\frac{-2(1+\gamma)}{1+5\gamma}}\right) dt^2 \tag{27}$$

This is the spherically symmetric metric with electromagnetic field in stationary space-time. We calculate the energy density ρ from equation (24), using the equation (25) as

$$8\pi\,\rho = \square\,\frac{2A}{1+5\gamma}\, r^{\frac{-4(1+3\gamma)}{1+5\gamma}} \tag{28}$$

The pressure is given by

$$8\pi\, p = -\frac{2A\gamma}{1+5\gamma}\, r^{\frac{-4(1+3\gamma)}{1+5\gamma}} \tag{29}$$

From equation (21), we obtain

$$F_{14} = \sqrt{\frac{A(1+3\gamma)}{1+5\gamma}}\; r^{\frac{-2(1+3\gamma)}{1+5\gamma}} \tag{30}$$

The value of the current density $\in$ is obtained from equation (6) as

$$4\pi \in = \frac{4\gamma}{1+5\gamma}\sqrt{\frac{A(1+3\gamma)}{1+5\gamma}}\left(1 - A\, r^{\frac{-2(1+\gamma)}{1+5\gamma}}\right)^{\frac{1}{2}} r^{\frac{-(3+11\gamma)}{1+5\gamma}} \tag{31}$$

We are assuming the equation of state condition $p = \gamma\rho$, where γ is arbitrary constant, in the study of the model. From the expression of energy density equation (28), it is observed that the energy density of the matter is always negative and hence our model (27) does not exist. Therefore in our further study, we are assuming the particular values of arbitrary constant γ in equation of state $p = \gamma\rho$ in the evolution of the model. It is seen that for $\gamma = 0$, $\gamma = 1/3$ and $\gamma = 1$, the energy density of the matter is always negative which shows the non – existence of dust universe, radiating universe and stiff dominated fluid universe. As we have assumed arbitrary constant γ in our equation of state $p = \gamma\rho$, we can take the negative values with γ in our study and it is realized that for $\gamma = -1$ and $\gamma = -1/3$, the energy density ρ of the fluid is positive and therefore we can evaluate the nature of model in these cases $\gamma = -1$ and $\gamma = -1/3$.

For $\gamma = -1$, we have $e^{\lambda} = (1 - A)^{-1}$, $e^{\nu} = (1 - A)$ and then metric is given as

$$d\tau^2 = -(1-A)^{-1} dr^2 - r^2 d\theta^2 - r^2 Sin^2\theta\, d\phi^2 + (1-A) dt^2 \tag{32}$$

The energy density ρ , the pressure p, the current density $\in$ and electromagnetic field F_{14} are given by

$$8\pi\, p = -8\pi\,\rho = -\frac{A}{2r^2}, \tag{33}$$

$$4\pi \in = \sqrt{\frac{A(1-A)}{2r^4}}, \tag{34}$$

$$F_{14} = \sqrt{\frac{A}{2r^2}} \tag{35}$$

The model (32) exists for $\gamma = -1$ with equation of state $p = -\rho$ and it represents dark energy model with vacuum energy, the most plausible and most puzzling dark energy candidate. The vacuum has a pressure equals to minus its energy density and this vacuum energy is mathematically equivalent to a cosmological constant providing the repulsive forces and opposing the gravitational pulls between the galaxies. It is noted that if A = 1, then the model (32) does not exist and it is singular. Further it is noticed that for any value of $A\ (\neq 1)$, the model is flat and thus equation (32) represents dark energy flat model. In particular, for A = 0, the model is flat and goes over to vacuum $(p = \rho = 0)$ and electromagnetic field gets disappeared in it.

For $\gamma = -1/3$, we write $e^{\lambda} = (1 - Ar^2)^{-1}$, $e^{\nu} = (1 - Ar^2)$ and the metric (1) reduces to

$$d\tau^2 = -(1 - Ar^2)^{-1} dr^2 - r^2 d\theta^2 - r^2 Sin^2\theta\, d\phi^2 + (1 - Ar^2) dt^2 \tag{36}$$

The physical parameters p, ρ, $\in$ and F_{14} are

$$8\pi\, p = -\frac{8\pi\,\rho}{3} = -A \tag{37}$$

$$4\pi \in = 0, \tag{38}$$

$$F_{14} = 0 \tag{39}$$

The scalar expansion Φ, rotation ω_{ij} and shear tensor σ_{ij} as given by Ellis [46] are

$$\Phi = u^{i}_{|i} , \tag{40}$$

$$\omega_{ij} = -\frac{1}{2}(u_{i|j} - u_{j|i}) - \frac{1}{2}(\dot{u}_i u_j - \dot{u}_j u_i) , \tag{41}$$

$$\sigma_{ij} = \frac{1}{2}(u_{i|j} + u_{j|i}) + \frac{1}{2}(\dot{u}_i u_j + \dot{u}_j u_i) - \frac{\Phi}{3}(g_{ij} + u_i u_j) \tag{42}$$

For the model (36), we have scalar expansion $\Phi = 0$ and rotation $\omega_{ij} = 0$. The shear tensor is given by

$$\sigma^2 = \frac{1}{2}\sigma_{ij}\sigma^{ij} = -\frac{A^2 r^2}{(1- Ar^2)} \tag{43}$$

The model (36) exists and it has negative pressure for equation of state parameter $\gamma = -1/3$. It represents dark energy model and it is free from electromagnetic field and current vanishes. For $A = 1/R^2$, the model (36) goes over to de-Sitter universe

$$d\tau^2 = -\left(1- \frac{r^2}{R^2}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 Sin^2\theta \, d\phi^2 + \left(1- \frac{r^2}{R^2}\right)dt^2 , \tag{44}$$

Of radius R and it is a dark energy model, if $A > 0$. Thus one can conclude that, there may be chance to have dark energy and dark matter in de-Sitter universe and it is fully occupied by dark energy and dark matter only, for $A > 0$. In particular for A = 0, our dark energy de-Sitter universe goes over to empty universe. Therefore, it is realized that the geometry of the de-Sitter universe is not only completely empty, but it is also governed by dark energy and dark matter which support the earlier work of Dhongle [44]. The equation of state $p = -\rho/3$ supports the model of dark energy star. In de-Sitter universe, the radius of the universe *R*, the energy density ρ and the cosmological constant Λ are related to each other by the equation

$$\frac{1}{R^2} = \frac{8\pi\rho+\Lambda}{3} \tag{45}$$

Comparing this with $A = 1/R^2 = 8\pi\rho/3$ it is realized that the cosmological constant Λ admits zero value. The cosmological constant Λ is one of the candidates of dark energy and responsible for accelerating expansion and in view of this, we would like to say that the model is neither expanding nor contracting since $\Lambda = 0$ and this is also supported by zero value of scalar expansion Φ. The model is shear less and without rotation.
The effect of magnetic field in this dark energy de-Sitter universe is automatically switched off. The nature of the model in relates with the geometry in the different coordinates transformation, the motion of test particles and the redshift of the spectrum is similar to the nature carried out in the work of Dhongle [44] and there is no any new geometrical and kinematical ideas which we are gaining with the assumption of electromagnetic field. The matching of our results with the work of Dhongle [44] may be due to the zero effect of magnetic field.

Conclusion

We have presented the solution of spherically symmetric model with electromagnetic field and with equation of state $p = \gamma\rho$, in which γ is arbitrary constant, by solving the Einstein's field equations in stationary space-time. It is observed that dust, radiating and stiff dominated spherically symmetric models does not exist. Spherically symmetric models for $\gamma = -1, -1/3$ exist and they represent dark energy models. Further it is seen that $\gamma = -1$ leads to dark energy flat model, which is singular at $A = 1$ and $\gamma = -1/3$ contributes dark energy de-Sitter universe for A > 0. Other geometrical and physical aspects of the model are also studied.

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