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Curious properties of odd and even numbered triangles

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Abstract
In this paper, I have explored two number triangles filled with odd and even natural numbers respectively. These simple looking triangles provides rich source of mathematical properties which are proved in this paper. In this connection, I have proved twelve results in this paper. These results will provide enough motivation to understand the pattern and beauty of numbers.

Keywords: odd numbered triangle, even numbered triangle, triangular numbers, sum of squares, rhombus property

Introduction
Among several number triangles that exists in mathematics, Pascal's Triangle is the most significant and prominent possessing enormously rich mathematical properties in it. In this paper, I shall consider two number triangles the first containing only odd natural numbers and second containing only even natural numbers and had proved few exciting new results. These new results will be motivating enough for further exploration in future.

Odd Numbered Triangle
Construction
Let us consider the following number triangle filled with consecutive odd natural numbers.

1
3 5
7 9 11
13 15 17 19
21 23 25 27 29
31 33 35 37 39 41
43 45 47 49 51 53 55
. . . .

Fig 1

Odd Numbered Triangle
Let n be a natural number representing the row number in the triangle of Figure 1. We notice that the triangle of Figure 1 is constructed using consecutive odd numbers such that row n has n + 1 numbers. Let $u_{n,m}$ denote the nth row, mth number of the odd numbered triangle in Figure 1. We call $u_{n,m}$ as the general term of odd numbered triangle.

We notice that the general term $u_{n,m}$ is given by $u_{n,m} = n(n-1)+(2m-1)$ (2.1) Where $1 \le m \le n$. Using (2.1), I will prove interesting results in the following sections.

Theorem 1

The sum of numbers in row n of odd numbered triangle is n^3. That is, $\sum_{m=1}^{n} u_{n,m} = n^3$ (2.2)

Proof: The required sum is given by

$$\sum_{m=1}^{n} u_{n,m} = \sum_{m=1}^{n} \left[n(n-1) + (2m-1) \right] = \sum_{m=1}^{n} \left(n^2 - n - 1 \right) + 2\sum_{m=1}^{n} m$$

$$= n\left(n^2 - n - 1 \right) + 2\left[\frac{n(n+1)}{2} \right] = n^3$$

This completes the proof.

Theorem 2

The alternate sum of numbers in row n of odd numbered triangle is n^2 if n is odd and $-n$ if n is even. That is,

$$\sum_{m=1}^{n} (-1)^{m-1} u_{n,m} = \begin{cases} n^2, \text{ if } n \text{ is odd} \\ -n, \text{ if } n \text{ is even} \end{cases} \quad (2.3)$$

Proof: First, we observe the following simple identity regarding alternate sum of first n natural numbers.

$1 - 2 + 3 - 4 + \cdots + (-1)^{k-1} k = \frac{k+1}{2}$ if k is odd (2.4) and

$1 - 2 + 3 - 4 + \cdots + (-1)^{k-1} k = -\frac{k}{2}$ if k is even (2.5) .

$$\sum_{m=1}^{n} (-1)^{m-1} u_{n,m} = \sum_{m=1}^{n} (-1)^{m-1} \left[n(n-1) + (2m-1) \right] = \sum_{m=1}^{n} (-1)^{m-1} \left[\left(n^2 - n - 1 \right) + 2m \right]$$

$$= \left(n^2 - n - 1 \right) \sum_{m=1}^{n} (-1)^{m-1} + 2\sum_{m=1}^{n} (-1)^{m-1} m \qquad (2.6)$$

If n is odd, then $\sum_{m=1}^{n} (-1)^{m-1} = 1$. Now using (2.4) in (2.6), we get

$$\sum_{m=1}^{n} (-1)^{m-1} u_{n,m} = \left(n^2 - n - 1 \right)(1) + 2\left(\frac{n+1}{2} \right) = n^2$$

If n is even, then $\sum_{m=1}^{n} (-1)^{m-1} = 0$. Now using (2.5) in (2.6), we get

$$\sum_{m=1}^{n} (-1)^{m-1} u_{n,m} = \left(n^2 - n - 1 \right)(0) + 2\left(-\frac{n}{2} \right) = -n$$

This completes the proof.

Theorem 3

The central numbers of odd numbered triangle are odd perfect squares.

That is, $u_{2k-1,k} = (2k-1)^2$ (2.7)

Proof: The middle term in the odd numbered row of odd numbered triangle in Figure 1 can be considered as central numbers of the triangle. Referring to Figure 1, such numbers will be of the form $u_{2k-1,k}$ for any natural number k. These numbers are shown in pink color in Figure 1.

Thus using (2.1), we have $u_{2k-1,k}=(2k-1)(2k-2)+(2k-1)=(2k-1)^2$

Since $(2k-1)^2$ are odd squares for any natural number *k*, this completes the proof.

Theorem 4 (Vertical Symmetry Sum)

For any row number *n*, and for any $1\le k\le n$ we have $u_{n,k}+u_{n,n-k+1}=2n^2$ (2.8)

Proof: From (2.1), we have $u_{n,k}+u_{n,n-k+1}=[n(n-1)+(2k-1)]+[n(n-1)+(2(n-k+1)-1)]=2n^2$

This completes the proof.

Theorem 5

The sum of all numbers among first *n* rows of odd numbered triangle is square of the *n*th triangular number. That is,

$$\sum_{k=1}^{n}\sum_{m=1}^{k}u_{k,m}=\left[\frac{n(n+1)}{2}\right]^2 \quad (2.9)$$

Proof: First we note that the *n*th triangular number is sum of first *n* natural numbers given by $\frac{n(n+1)}{2}$. Now using (2.2), the required sum is given by $\sum_{k=1}^{n}\sum_{m=1}^{k}u_{k,m}=\sum_{k=1}^{n}k^3=\left[\frac{n(n+1)}{2}\right]^2$

This completes the proof.

Theorem 6 (Rhombus Property)

The four numbers located in (*n* – 1)th, *n*th and (*n* + 1)st rows of odd numbered triangle satisfy

$$u_{n,m}\times u_{n,m+1}-u_{n-1,m}\times u_{n+1,m+1}=2\left[u_{n+1,m+1}-2(2m+1)\right] \quad (2.10)$$

Proof: Using (2.1) and simplifying we have

$$\begin{aligned}u_{n,m}\times u_{n,m+1}-u_{n-1,m}\times u_{n+1,m+1}&=\left[\left(n(n-1)+(2m-1)\right)\times\left(n(n-1)+(2m+1)\right)\right]\\&\quad-\left[\left((n-1)(n-2)+(2m-1)\right)\times\left((n+1)n+(2m+1)\right)\right]\\&=2\left[n(n+1)-(2m+1)\right]=2\left[u_{n+1,m+1}-2(2m+1)\right]\end{aligned}$$

This completes the proof.

Even Numbered Triangle

Construction

Let us consider the following number triangle filled with consecutive even natural numbers.

2
4 6
8 10 12
14 16 18 20
22 24 26 28 30
32 34 36 38 40 42
44 46 48 50 52 54 56
. . . .

Fig 2

Even Numbered Triangle

Let *n* be a natural number representing the row number in the triangle of Figure 2. We notice that the triangle of Figure 2 is constructed using consecutive even numbers such that row *n* has *n* + 1 numbers. Let $v_{n,m}$ denote the *n*th row, *m*th number of the even numbered triangle in Figure 2. We call $v_{n,m}$ as the general term of even numbered triangle.

We notice that the general term $v_{n,m}$ is given by $v_{n,m} = n(n-1) + 2m \quad (3.1)$ where $1 \le m \le n$. Using (3.1), I will prove interesting results in the following sections.

Theorem 7

The sum of numbers in row *n* of even numbered triangle is $n^3 + n$. That is, $\sum_{m=1}^{n} v_{n,m} = n^3 + n \quad (3.2)$

Proof: The required sum is given by

$$\sum_{m=1}^{n} v_{n,m} = \sum_{m=1}^{n} \left[n(n-1) + 2m \right] = \sum_{m=1}^{n} \left(n^2 - n \right) + 2\sum_{m=1}^{n} m$$

$$= n\left(n^2 - n \right) + 2\left[\frac{n(n+1)}{2} \right] = n^3 + n$$

This completes the proof.

Theorem 8

The alternate sum of numbers in row *n* of even numbered triangle is n^2+1 if *n* is odd and $-n$ if *n* is even. That is,

$$\sum_{m=1}^{n} (-1)^{m-1} v_{n,m} = \begin{cases} n^2 + 1, & if \ \ n \ is \ odd \\ -n, & if \ \ n \ is \ even \end{cases} \qquad (3.3)$$

Proof: Using (3.1), we have

$$\sum_{m=1}^{n} (-1)^{m-1} v_{n,m} = \sum_{m=1}^{n} (-1)^{m-1} \left[n(n-1) + 2m \right] = \left(n^2 - n \right) \sum_{m=1}^{n} (-1)^{m-1} + 2\sum_{m=1}^{n} (-1)^{m-1} m \qquad (3.4)$$

If *n* is odd, then $\sum_{m=1}^{n} (-1)^{m-1} = 1$. Now using (2.4) in (3.4), we get

$$\sum_{m=1}^{n} (-1)^{m-1} v_{n,m} = \left(n^2 - n \right)(1) + 2\left(\frac{n+1}{2} \right) = n^2 + 1$$

If *n* is even, then $\sum_{m=1}^{n} (-1)^{m-1} = 0$. Now using (2.5) in (3.4), we get

$$\sum_{m=1}^{n} (-1)^{m-1} v_{n,m} = \left(n^2 - n - 1 \right)(0) + 2\left(-\frac{n}{2} \right) = -n$$

This completes the proof.

Theorem 9

The central numbers of even numbered triangle are one more than odd perfect squares.

That is, $v_{2k-1,k} = (2k-1)^2 + 1 \quad (3.5)$

Proof: The middle term in the even numbered row of odd numbered triangle in Figure 2 can be considered as central numbers of the triangle. Referring to Figure 2, such numbers will be of the form $v_{2k-1,k}$ for any natural number *k*. These numbers are shown in pink color in Figure 2.

Thus using (3.1), we have $v_{2k-1,k} = \left[\left(2k-1\right)\left(2k-2\right)+2k\right]+(-1+1)=(2k-1)^2+1$

Since $(2k-1)^2$ are odd squares for any natural number *k*, this completes the proof.

Theorem 10 (Vertical Symmetry Sum)

For any row number *n*, and for any $1 \le k \le n$ we have $v_{n,k} + v_{n,\mathrm{n}-k+1} = 2(n^2+1)$ (3.6)

Proof: From (2.1), we have $v_{n,k} + v_{n,n-k+1} = \left[n(n-1)+2k\right]+\left[n(n-1)+2(n-k+1)\right] = 2(n^2+1)$

This completes the proof.

Theorem 11

The sum of all numbers among first *n* rows of even numbered triangle is sum of *n*th triangular number and its square. That is,

$$\sum_{k=1}^{n}\sum_{m=1}^{k} v_{k,m} = \frac{n(n+1)}{2} + \left[\frac{n(n+1)}{2}\right]^2 \quad (3.7)$$

Proof: First we note that the *n*th triangular number is sum of first *n* natural numbers given by $\frac{n(n+1)}{2}$. Now using (3.2), the required sum is given by

$$\sum_{k=1}^{n}\sum_{m=1}^{k} v_{k,m} = \sum_{k=1}^{n}(k^3+k) = \sum_{k=1}^{n} k + \sum_{k=1}^{n} k^3 = \frac{n(n+1)}{2} + \left[\frac{n(n+1)}{2}\right]^2$$

This completes the proof.

Theorem 12 (Rhombus Property)

The four numbers located in (*n* – 1)th, *n*th and (*n* + 1)st rows of even numbered triangle satisfy $v_{n,m} \times v_{n,m+1} - v_{n-1,m} \times v_{n+1,m+1} = 2\left[v_{n+1,m+1} - 4(m+1)\right]$ (3.8)

Proof: Using (3.1) and simplifying we have

$$\begin{aligned} v_{n,m} \times v_{n,m+1} - v_{n-1,m} \times v_{n+1,m+1} &= \left[\left(n(n-1)+2m\right)\times\left(n(n-1)+(2m+2)\right)\right] \\ &\quad -\left[\left((n-1)(n-2)+2m\right)\times\left((n+1)n+(2m+2)\right)\right] \\ &= 2\left[n(n+1)-2(m+1)\right] = 2\left[v_{n+1,m+1} - 4(m+1)\right] \end{aligned}$$

This completes the proof.

Conclusion

By considering two types of triangles in sections 2 and 3 respectively, I had proved 12 results in this paper. In particular, in section 2, I had constructed odd numbered triangle in Figure 1, which consists of all odd numbers in which row *n* contains *n* + 1 consecutive odd numbers. Using this triangle, I had proved six interesting properties in which results established in theorems 2, 4, 5 and 6 are new. Similarly in section 3, I had constructed even numbered triangle in Figure 2, which consists of all even numbers in which row *n* contains *n* + 1 consecutive even numbers. Using this triangle, I had proved six formulas in which the results established in theorems 8, 10, 11 and 12 are new. Since the four entries of odd numbered and even numbered triangles in theorems 6 and 12 forms a Rhombus configuration, the formulas established in (2.10) and (3.8) are known as Rhombus property.

It will also be interesting to note that the row sum beginning from third row of even numbered triangle (in Figure 2) is exactly twice the magic sum obtained for a $n \times n$ magic square consisting of first n^2 natural numbers. The twelve properties established in this paper will exhibit the beauty behind the arrangement of odd and even numbers in the respective triangles and provide scope for exploring more interesting properties.

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