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A review article on fixed point theory & its application use in economics & game theory

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Abstract
In this manuscript, we provide an introduction to those topics of fixed point theory that is one of the most important and powerful tools of the modern mathematics not only it is used on a daily basis in pure and applied mathematics but it is also solving a bridge between analysis and topology and provide a very fruitful are of interaction between the two. The theory of fixed points belongs to topology, a part of mathematics created at the end of the nineteenth century. The famous French Mathematician H. Poincare (1854-1912) was the founder of the fixed point approach. He had deep insight into its future importance for problems of mathematical analysis and celestial mechanics and took an active part in its development. In spite of their elementary character, the results given here have a number of significant applications. Some of these are presented at the end of the chapter.

Keywords: fixed point, game theory, brouwer's theorem, kakutani's theorem

Introduction

Fixed point theory is a rich, interesting and exciting branch of mathematics. It is a relatively young but fully developed area of research. Study of the existence of fixed point's falls within several domains such as classical analysis, functional analysis, and operator theory, general and algebraic topology. Fixed points and fixed point theorems have always been a major theoretical tool in fields as widely apart as topology, mathematical economics, game theory, and approximation theory and initial and boundary value problems in ordinary and partial differential equations.

The scientific basis of the fixed point theory was established in the 20th century. The fundamental result of this theory is the Picard-Banach-Caccioppoli contraction principle (from the '30s), which generated important lines of research and applications of the theory to functional equations, differential equations, integral equations, etc.

Classic theorems of this theory are the theorems of Tarki, Bourbaki, Banach, Perov, Luxemburg-Jung, Brower, Schauder, Tihonov, and Brouwder-Ghode-Kirk (Rus, Petruşel, Petruşel, 2008).

Banach's fixed point theory, also known as the contraction principle, is an important tool in the theory of metric spaces. It guarantees the existence and uniqueness of solutions to equations of the form x = f(x), for a wide range of applications f, and it also provides a constructive method to determine these solutions.

In this paper, the basic results of fixed point theory valuable to the economic researches are reviewed. The primary goal of this paper is to present Brouwer's and Kakutani's theorems in order to analyse potential applications in the field of economic research.

Moreover, recently, the usefulness of this concept for applications increased enormously by the development of accurate and efficient techniques for computing fixed points, making fixed point methods a major tool in the arsenal of mathematics Fixed point theory is equivalent to best approximation, variation inequality and the maximal elements in mathematical economics. The sequence of iterates of fixed point theory can be applied to find a solution of a variation inequality and the best approximation theory. The theory of fixed points is concerned with the conditions which guarantee that a map T: X → X of a topological space X into itself. Admits one or more fixed points that are points x in X for which x = Tx. For example, a translation, i.e. the mapping T (x) = x + a for a fixed a, has no fixed point, a rotation of the plane has a single fixed point (the centre of rotation), the mapping $x \to x^2$ of R into itself has two fixed points (0 and 1)

Fixed point theory examines the existence of the point x belonging to the domain of function f for which stands that f(x) = x, i.e. function values are equivalent to identical function mapping. In Figure 1 three intersections of function f(x) and function y = x represents the fixed points. A more subtle analysis would lead to the conclusion that a marginal change in the f(x) function causes additional fixed points to emerge.

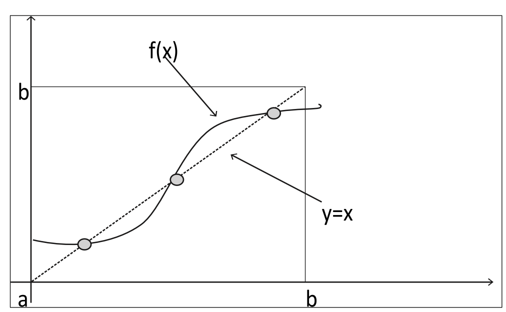

Fig 1: Three fixed points of the function f

If a certain function g is presented as g(x) = f(x) – x, then the solution to the equation g(x) = 0 is the fixed point of the function f (see Figure 2).

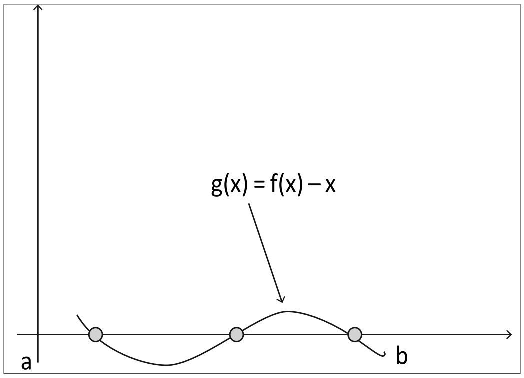

Fig 2: Solution to the equation g(x) = 0 is the fixed point of function f

Fixed point theory is applied in different scientific fields. In mathematics, it is used for solving different equations, creating approximations and simulations, in game theory, etc. In the field of economics, it is often used in the process of determining the coincidence point of supply and demand functions. Actually, fixed points (i.e. equilibriums) are at the core of many generic economic models. This theory enhanced the understanding of many other problems inherent to economic models such as comparative statics, robustness to marginal changes and equilibrium stability as well as equilibrium calculation. The earliest fixed point theorem is that of Brouwer [11], who in (1912), proved that a continuous self-mapping T of the closed unit ball Rn has at least one fixed point, that is, a point x such that Tx = x. Several proof of this historic result can be found in the existing literature. Another fundamental result after Brouwer's fixed point theorem was given by polish mathematicians S. Banach in (1922). Banach proved a theorem, which ensures under appropriate conditions, the existence and uniqueness of a fixed point. This result is popularly known as "Banach fixed point theorems" or the "Banach Contraction Principle". It states that a contraction mapping of complete metric space into itself has a unique fixed point. It is the simplest and one of the most versatile results in fixed point theory. Being based on an iteration process, it can be implemented on a computer to find the fixed point of a contractive map, it produces approximations of any required accuracy. Due to its applications in various disciplines of mathematics and mathematical sciences, the Banach contraction principle has been extensively studied and generalized on many settings and fixed point theorems have been established. There are large classes of mappings for which fixed point theorems have been studied. It includes contractive mappings, contraction of various order mappings, Ciric contraction, asymptotically regular, densifying etc. Apart from single mappings, pair of mappings, the sequence of mappings and family of mappings are also some of the classes of mappings that have interested mathematicians.

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Chu-Diaz [18] and Bryant [14]

T^n to be a contraction in order to get a unique fixed point. Rakotch [78] and Boyd-Wong [10] have attempted to generalize the Banach contraction principle by replacing the Lipschitz constant by some real valued function whose value is less than one. But generally, in order to accommodate a variety of continuous and discontinuous functions, attempts were made to replace the contractive conditions by some general form of mapping condition (called generalized contraction). In (1977), Rhoades [80] has made a comprehensive study and compared various contractive conditions which are scattered in the literature. He also introduced some new definitions. In (1978), Rhoades enlarged his system, where a number of definitions were duplicated by Fisher work [22, 23]. In (1980), Hegedus [32] defined the concept of generalized Banach contraction. In it appeared the diameter of a non-finite set in the inequality of definition, for the first time. This new definition was the base of many generalizations. These definitions were systematized by Park [68]. In 1992, Meszaros [54] proved the equivalence of forty-three contractive definitions and many inclusion relations. Recently, Rhoades [82] tried to obtain general results from which a lot of already known results follow as a corollary. He also obtained some new results. On the other hand, in (1976), Jungck [36] generalized the Banach contraction principle by introducing a contraction condition for a pair of commuting self mapping son metric space and pointed out the potential of commuting mappings for generalizing fixed point theorems in metric spaces [37]. Jungck's results have been further generalized by considering the general type of contractive conditions on the pair of mappings by Das and Naik [19], Kasahara [44], Park [68, 69, 70], Ranganathan [79], Singh [95] and several others. Further generalizations have also been obtained by taking contractive type conditions for three self mappings on a metric space – one of the mappings commuting with other two by Qureshi and Awadhiya [76], Bhola and Sharma [8], Khan and Imdad [48], Fisher [26], etc. In (1982) Sessa [88], initiated the tradition of improving commutativity conditions in metrical common fixed point theorems. While doing so Sessa introduced the notion of weak commutativity. Motivated by Sessa [88], Jungck [38] defined the concept of compatibility of two mappings, which includes weakly commuting mappings as a proper subclass. After this definition there is a multitude of compatibility like conditions such as: compatibility of type (A) (Jungck, Murthy & Cho [39], Compatibility of type (B) (Pathak and Khan [65]), compatibility of type (P) (Pathak *et al*. [66]), weak compatibility of type (A) (Lal, Murthy & Cho [52]), p-weak compatibility (Ume and Kim [110]) whose details can be seen in their introducing papers. In (1998), Jungck and Rhoades [40] termed a pair of mappings to be weakly compatible (or coincidently commuting) if they commute at their coincidence point. In (2001), Ahmed and Rhoades [2] producer some common fixed point theorems for compatible mappings on complete metrically convex metric spaces thereafter in (2002) Aamir and Moutawaki [1] gave some new fixed point theorems under strict contractive conditions. In (2003) Som [108] obtained some common fixed point results for a weaker type of mappings than commuting or weakly commuting, called compatible mappings, satisfying a more general inequality condition. In (2003) Phaneendra [72] have obtained common fixed point theorems for a pair of self maps which commute at their coincidence points, called weakly compatible maps using the idea of an orbit relative to self-maps. It has been known since the paper of Kannan [42] that there exists a map possessing discontinuity in their domain but still admitting fixed points. However, in every case, the maps involved were continuous at the fixed point. Recently some authors attempted to relax continuity requirement in such results and for the work of this kind one May refer to Pant [61, 62, 63], Singh and Mishra [103] and Pant, Lohari and Jha [64]. In (2001) Beg [5] proved an iteration scheme for asymptotically non expansive mappings in convex metric spaces. In (2003) he [6] obtained an iteration process for non process for non-linear mappings in uniformly convex linear metric spaces. He also proved in [7] fixed point set function of non expansive random mapping on metric spaces. In (2003) Fisher & Duran [25] proved some fixed point theorems for multivalued mappings or orbitally complete uniform spaces.

In (2003), Suzuki [109] generalized the result of Kanan [43]. In (2003), Popa [73] has improved the result of several authors by removing the assumption of continuity, relaxing compatibility to the weak compatibility property and replacing the completeness of the spaces with a set of four alternative conditions for four functions satisfying and implicit relations. In (2003) Proinov [75] established the Meir-Keeler type contractive conditions and the contractive definitions involving gauge functions.

The concept of 2-metric spaces has been investigated initially by Gahler [27]. This concept was subsequently enhanced by Gahler [28, 29], White [112] and several others. On the other hand Iseki [33], Iseki-Sharma-Sharma [34], Khan-Fisher [47], Khan [46], Singh-Tiwari-Gupta [98] and a number of other authors have studied the aspects of fixed point theory in the setting of 2-metric space.

Khan [45], Murthy-Chang-Cho-Sharma [56], Rhoades [83], Singh-Tiwari and Gupta [98] and Naidu-Prasad [57] introduced the concepts of weakly commuting pairs of self mappings, compatible pairs of self mappings of type (A) in a 2-metric spaces, and they have proved several fixed point theorems by using the weakly commuting pairs of self mappings, compatible pairs of self mappings of type (A) in a 2-metric spaces. In (2001), Naidu [58] has proved some fixed point theorems for pairs as well as quadruples of self maps on a 2-metric space satisfying certain generalized contraction condition. In (2001) B. Singh and R.K. Sharma have proved some common fixed point theorems using the concept of compatible mappings in 2-metric spaces.

Rhoades *et al*. [84] introduced the concept of relative asymptotic regularity for a pair of mapping on a metric space and Jungck [38] proposed the concept of compatible mappings and weakly commuting mappings. Sessa [88] and others used both cited concepts and gave many interesting results.

In (1979) Singh [106] improved the results of Brosowski [12], using a fixed point theorem of Jungek [36], Sahab, Khan and Sessa [87] generalized the results of Singh [106]. Pathak, Cho and Kang [67] gave an application of Jungck's [41] fixed point theorem to best approximation theory. They extended the results of Singh [106] and Sahab *et al*. [87]. In (2001) Change [20] generalized the result of Sahab *et al*. [87] in best approximation theory under some weaker conditions. In (2003), Nashne [59] proved some fixed point theorems without star-shaped ness condition of domain and linearity condition of mappings in setup domain and linearity condition of mapping in the setup of normed linear space. Recently Vijayraju and Marudai [111] proved some results on common fixed compact mappings are established in the setting of normed linear space which is an extension of results of Sahab, Khan [86] and Dotson [21] as a consequence some applications of best approximations are established.

Applications of the fixed point theorems

Fixed point theorems have numerous applications in mathematics. Most of the theorems ensuring the existence of solutions for differential, integral, operator or other equations can be reduced to fixed point theorems. They are also used in a new area of mathematical applications eg. In mathematical economics, game theory, approximation theory, dynamic programming and solutions of non-linear integral equations.

The economic application of Brouwer's and Kakutani's theorems

Fixed point theorems are most frequently used for proving that at least one equilibrium exists in an economic or game theory model. Equilibrium is the vector of endogenous model variables when all agents are presumed to act rationally, through utility maximization, and when an individual agent regards all other endogenous variables ceteris paribus.

Application 1. Let P be the price and Q the quantity. Let P=D(Q) be the demand function and P=S(Q) supply function. If supply is equal to demand then there exists market equilibrium, presented with equilibrium [Q*, P*] (Q* being the equilibrium quantity and P* being the equilibrium price, see Figure 5).

Market price differs from equilibrium price due to effects of competition. That is why a market is regarded as stable when price converges to equilibrium price

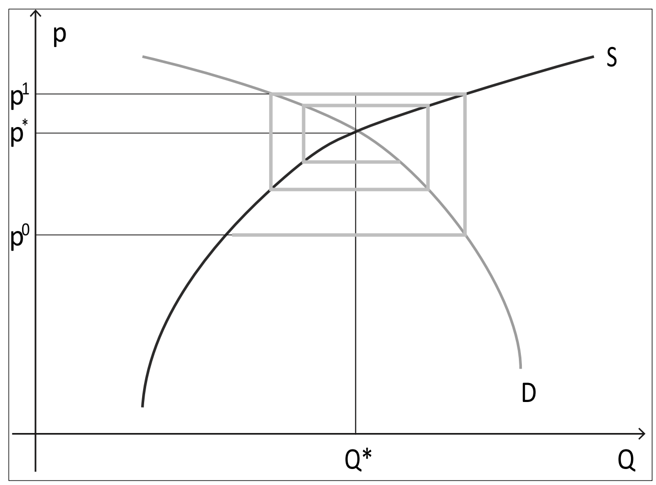

Fig 3: The equilibrium

Let P min be the lowest price for a commodity in the given market. Let Pmax be the highest price at which a commodity can be sold in the given market. Observe the following function:

$$f:[P\text{min}, P\text{max}] \rightarrow [P\text{min}, P\text{max}]$$

defined with:

$$f(P) = D(S^{-1}(P_0)),$$

Function f is adequately defined for a given price domain because the monotony of demand and supply function allows for the existence of adequate inverse functions. It is straightforward to prove that function is continuous. Since domain [Pmin, Pmax] is compact and convex, it can be concluded that fixed point (price) exists on the basis of Brouwer's theorem.

Let us describe the algorithm used in order to determine equilibrium price. Let P0 be the market price which is lower than equilibrium price, i.e. P0 < P*. Let Q0D and Q0S be the demand quantity and supply quantity, respectively for price P0. The following is true then: Q0D = D-1(P0) and Q0S = S-1(P0).
Given that D is monotonic decreasing function and D is monotonic increasing function then inverse functions D-1 and S-1 exist. If producers increase the price to P1 (for a demanded quantity) then the following is true: P1 = D(Q0S) = D(S-1(P0)), and P1 >P*. The following stands for corresponding demand and supply quantities Q1D and Q1:

Q1D > Q1S,

Which leads to deviation of | Q1S– Q1D|. If producers decrease the price to P2 so that:

P2 = D(Q1S) = D(S-1(P1)).

If we repeat this algorithm, we get the sequence of the prices P0, P1, P2, … Pk,… for which: Pk = D(S-1(Pk-1)), k =1,2,...
According to that the sequence of the prices (Pk) converges to the equilibrium price P*. This is presented in Figure 5. Meznik [7] has also considered this application.
Application 2 (Nash equilibrium). Let N be a fixed finite set, which is called "set of players (participants)". Each player is labeled with index i.
Normal-form game is an ordered triple, in which for every i ∈ N, Si is non-empty sets, and ui is functions ui: iΠ∈N Si → R. We will regard Si as a set of strategies, and i as a user's gain (utility) function (i∈N). If we denote SN = iΠ∈N Si, then every s∈SN is the outcome (strategic profile) in the game Γ. Player i chooses strategy si∈Si. When all players choose their strategies, then the outcome of game s and gain for every player i − ui(s).
From the aforementioned the single normal-form game is defined when the following three elements are defined:

1. set of game participants,
2. set of strategies for each player,
3. gain function for each player.

Firstly, several useful notations will be introduced. Let s = (s1, s2,..., sn) be a strategic profile. Then:

s–i = (s1, s2,..., si–1, si+1,..., sn)

(s–i, s*i) = (s1, s2,..., si–1, s*i, si+1,..., sn)

Nash equilibrium is the strategic profile s*∈S in which for every i ∈ N stands that ui(s*–i, s*i) ≥ ui(s*–i, si) for si∈Si.
Nash theorem [8, 9]. If strategic sets of each player are non-empty, convex and compact and their utility functions are continuous and quasiconcave for s–i then Nash equilibrium exists for a normal-form game.
The proof of this theorem is implied by Kakutani's theorem since the best answer function is defined with bi(s–i) = arg max {ui(si, s–i)|si∈Si} and $b(s) = \prod_{i=1}^{n} b_i(s_{-i})$. Function b is well defined on the basis of Weierstrass theorem. It should be noticed that if s*∈b(s*) then s*i∈b(s*–i) for every i∈N, which leads to the conclusion that s* is Nash equilibrium.

Application 3 (Cournot oligopoly, see [4]). Cournot oligopoly model is the model for which holds the next assumptions: • there are n firms;
a firm i produces commodity i for i∈{1,2,…,n}(qi ≥ 0 is the quantity of commodity and pi is the price);
all goods (commodities) are perfectly divisible;
the goal of each firm is to choose an amount of product that maximizes its own profit given the production levels chosen by other firms.
Let q–i = (q1,..., qi–1, qi+1,...,qn) be a vector of quantities produced by the other firms. We can assume that: pi = Pi (qi, qi–1) = ai – biqi + j∑≠i dij qj, i =1,..., n
i.e. price pi is decreasing in its own quantity qi and, due to complementarities between the commodities, is assumed to be increasing in the quantities qj, j ≠ i, of the other firms (parameters ai, bi, dij are positive).

Fixed Point Theory Applied to the Game Theory. Particular Case – Games for the Field of Quality

The problems related to the analysis of the quality of a tangible or intangible product may be approached, in some cases, as problems from the game theory.
Creating a tangible or intangible product depends on two groups of factors: one group that increases the values of the product's quality indicators, and the other group that decreases the values of the product's quality indicators. Therefore, the first player is determined by the factors that increase the values of the quality indicators, and the second player by the other group of factors. The first player "wishes" to create a high quality product, whereas the second player "wishes" to create a poor quality product. The result of the competition between them is the actual quality of the product. According to dedicated scientific notation, we denote by A1 = {α1,..., αi,...,αm}the set of factors that lead to an increase in the values of the quality indicators, and by A2 = {a1,..., aj,...,an} the set of factors that lead to a decrease. At one moment of time from the life cycle of a tangible or intangible product, each player has a certain influence on the values of the quality indicators. Each player chooses an action αi from A1 and aj from A2. The actions refer to the effect of factor i on the values of the quality indicators. The utility of choosing action αi by the first player can be described mathematically by a real function f1 (αi, aj) and its values can be interpreted as a win for the first player, in this situation. Function f2 (αi, aj) represents the second player's loss, in this situation. According to the specialty literature, the fact that the sum of the game is null can be written as (Owen, 1974):

f1(αi, aj) + f2(αi, aj) = 0.

The question is how the first player can choose the action αi in order to achieve a maximum gain f1(αi, aj), knowing that the other player has the same objective (the term utility was introduced by von Neumann, and it significantly expanded the concept of "game", suggesting that a "result" of a game is not only something financial, but also a diverse multitude of events for which each player shows interest, quantified by their utility). The Nash Equilibrium in a pure strategy is represented by a strategic profile in which the strategy of each player is the best response to the strategy chosen by the other player. Thus, the conflict situations regarding creating tangible or intangible products of a high quality level and their management can be modelled by using the interconnections between these theories (Figure no.4):

Fig 4: Illustrating the interconnection between the theories

Several authors (i.e. Yang and Yu, 2002; Yu and Yang, 2004; Lin, 2005) have demonstrated that the fixed point theory can be applied to optimisation problems, game theory problems, and also in problems related to the Nash equilibrium.

Conclusions

The fixed point theory has had many applications in the last decades. Its applications are very useful and interesting to the optimisation theory, to the game theory, to conflict situations, but also to the mathematical modelling of quality and its management. This paper is the starting point for further research on the fixed point theory application in economics. In order to clarify the potential scope of utilization for economic research purposes elementary topics of fixed point theory are hereby introduced. Brouwer's and Kakutani's theorems reviewed in this paper are the basis for further analysis and assessment of equilibrium. Although its application is a great challenge, this theory draws attention of mathematicians all around the world. Authors of this paper intend to apply this theory to the research of topics such as market concentration and competition as well as the determination of equilibrium states.

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