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Ramanujan summation for geometric progressions

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Abstract
Among several summation methods that exist in mathematics, Indian mathematician Srinivasa Ramanujan introduced a novel way of determining sum of divergent series related to Riemann zeta function. In this paper, I extended the idea of Ramanujan summation for Geometric Progressions whose common ratio is greater than 1. In doing so, a general and new result was established in this paper. Using this general result, I had obtained Ramanujan summation of two Geometric Progressions and had also explained the geometric meaning of the answers obtained.

Keywords: ramanujan summation, divergent series, definite integral, geometric progression, common ratio

Introduction

The concept of Ramanujan Summation (abbreviated as *RS*) was introduced by Srinivasa Ramanujan during the beginning of 20th century. The purpose of this summation method was to relate it to Riemann Zeta function which was revealed only lately by English mathematicians Littlewood and Hardy. Ever since, Ramanujan summation methods were identified, several generalizations and alternate techniques were employed by mathematicians throughout the globe. This paper is written for the purpose of determining Ramanujan summation for Geometric progressions whose common ratio is greater than 1.

Definitions

Let a, r be any two non-zero real numbers. Then the sequence of terms $a, ar, ar^2, ar^3, ar^4, ar^5, \ldots$ (2.1) is said to form a Geometric Progression abbreviated as G.P. The number a is called the first term and r is called the common ratio of the G.P. In this paper, I will consider Geometric progressions whose common ratio $r > 1$.

Let $\{a_n\}_{n=0}^{\infty}$ be any sequence of real numbers. Let S_n be sum of first n terms of the sequence $\{a_n\}_{n=0}^{\infty}$. The Ramanujan summation (see [1]) of the divergent series $\sum_{n=0}^{\infty} a_n$ is defined as

$$(RS)\left(\sum_{n=0}^{\infty} a_n\right) = \int_{n=-1}^{0}\left(\sum_{k=0}^{n-1} a_k\right)dn = \int_{n=-1}^{0} S_n\, dn \quad (2.2)$$

Theorem 1

The Ramanujan summation of the terms of the Geometric Progression (G.P.) as defined in (2.1) is given by

$$(RS)\left(\sum_{n=0}^{\infty} ar^{n-1}\right) = (RS)(a + ar + ar^2 + ar^3 + \cdots) = a\left[\frac{1}{r\log_e r} - \frac{1}{r-1}\right], r > 1 \quad (3.1)$$

Proof. First, notice that the nth term of the G.P. whose terms are $a, ar, ar^2, ar^3, ar^4, ar^5, \ldots$ is ar^{n-1}. If S_n is sum of first n terms of this G.P., then

$$S_n = \sum_{k=0}^{n-1} ar^k = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(r^n - 1)}{r-1} \quad (3.2)$$

Since $r > 1$, S_n is well defined and the series $a + ar + ar^2 + ar^3 + \cdots$ is divergent. Now by (2.2) and (3.2), we get

$$(RS)\left(\sum_{n=0}^{\infty} ar^{n-1}\right) = (RS)(a + ar + ar^2 + ar^3 + \cdots) = \int_{n=-1}^{0} \frac{a(r^n - 1)}{r-1} dn$$
$$= \frac{a}{r-1}\left(\frac{r^n}{\log_e r} - n\right)_{n=-1}^{0} = \frac{a}{r-1}\left[\frac{1}{\log_e r}\left(1 - \frac{1}{r}\right) - 1\right]$$
$$= a\left[\frac{1}{r\log_e r} - \frac{1}{r-1}\right]$$

This completes the proof.

Ramanujan Summation for Few Geometric Progressions Corollary

$$(RS)(1+2+4+16+32+64+\cdots) = -0.27865 \ (4.1)$$

$$(RS)(3+12+48+192+\cdots) = -0.458989 \ (4.2)$$

Proof: Using (3.1) of theorem 1, and considering a = 1, r = 2 we get

$$(RS)(1+2+4+8+16+32+64+\cdots) = \frac{1}{2\log_e(2)} - 1 = -0.27865$$

This proves (4.1). Now to prove (4.2), consider a = 3 and r = 4. Using (3.1) of theorem 1, we get

$$(RS)(3+12+48+192+\cdots) = 3 \times \left[\frac{1}{4\log_e(4)} - \frac{1}{3}\right] = -0.458989$$

This proves (4.2) and hence completes the proof.

Geometric Meaning

In this section, I will provide the geometric meaning behind the answers obtained in the corollary established in equations (4.1) and (4.2).

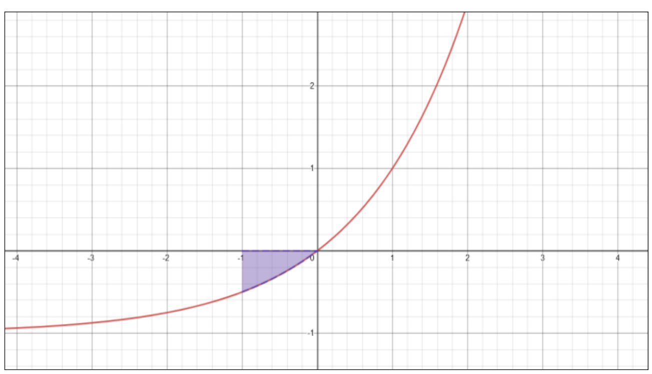

Fig 1: Area bounded by $f(x) = 2^x - 1$ between X – axis and $[-1, 0]$

In view of (3.2), for obtaining (4.1), we see that $S_n = 2^n - 1$. Hence the Ramanujan summation for the series $1+2+4+16+32+64+\cdots$ is given by

$$(RS)(1+2+4+16+32+64+\cdots) = \int_{n=-1}^{0} \left(2^n - 1\right) dn = -0.27865$$

From Figure 1, we notice that the region bounded by the function $f(x) = 2^x - 1$ between X – axis in $[-1, 0]$ lies below X – axis as shown in shaded portion of Figure 1. This explains the negative answer that we obtained in (4.1).

Similarly, in view of (3.2), for obtaining (4.2), we see that $S_n = 4^n - 1$. Hence the Ramanujan summation for the series $3+12+48+192+\cdots$ is given by

$$(RS)(3+12+48+192+\cdots)=\int_{n=-1}^{0}\left(4^{n}-1\right)dn=-0.458989$$

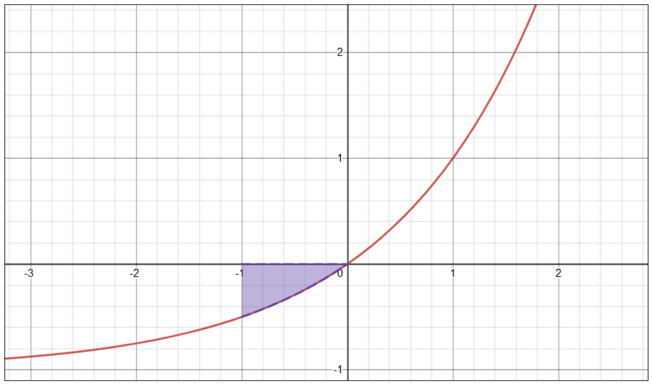

Fig 2: Area bounded by $f(x)=4^{x}-1$ between X – axis and $[-1,0]$

From Figure 2, we notice that the region bounded by the function $f(x)=4^{x}-1$ between X – axis in $[-1,0]$ lies below X – axis as shown in shaded portion of Figure 2. This explains the negative answer that we obtained in (4.2).

Conclusion

The concept of Ramanujan summation was to assign a particular value for divergent series of real numbers. This idea is similar to Cesaro summation. In this paper, I had proved a new result concerning determining Ramanujan summation for Geometric progressions whose common ratio is greater than 1 in (3.1) of theorem 1. The choice of common ratio greater than 1 is made to ensure that the geometric series is divergent which enables us to determine Ramanujan summation.

Through the more general result obtained in equation (3.1) of theorem 1, I had obtained Ramanujan summation for two particular Geometric progressions in which the common ratios are 2 and 4 respectively. The computed values agree with the geometric illustrations provided in section 4.2.

Similarly, using (3.1) we can determine Ramanujan summation of several Geometric progressions whose common ratio is greater than 1. These new results obtained will provide more ideas and allow for generalizing further in the landscape of Ramanujan summation methods.

References

1. Sivaraman R. Understanding Ramanujan Summation, International Journal of Advanced Science and Technology,2020:29(7):1472-1485.
2. Sivaraman R. “Sum of powers of natural numbers”, AUT AUT Research Journal,2020:11(4):353-359.
3. Ramanujan S, Manuscript Book 1 of Srinvasa Ramanujan, First Notebook, Chapter, 7, 66-68.
4. Bruce C Berndt, Ramanujan’s Notebooks Part II, Springer, Corrected Second Edition, 1999.
5. Hardy GH, Littlewood JE. Contributions to the theory of Riemann zeta-function and the theory of distribution of primes, Acta Arithmetica,1916:41(1):119-196.
6. Plouffe S. Identities inspired by Ramanujan Notebooks II, part 1, July 21 (1998), and part 2, 2006.
7. Bruce C Berndt. An Unpublished Manuscript of Ramanujan on Infinite Series Identities, Illinois University, American Mathematical Society publication.
8. Sivaraman R. “Remembering Ramanujan”, Advances in Mathematics: Scientific Journal,2020:9(1):489-506.
9. Sivaraman R. Bernoulli Polynomials and Ramanujan Summation, Proceedings of First International Conference on Mathematical Modeling and Computational Science, Advances in Intelligent Systems and Computing, Vol. 1292, Springer Nature, 2021, 475-484.
10. Candelpergher B, Gopalakrishna Gadiyar H, Padma R. Ramanujan Summation and the Exponential Generating Function, Cornell University, 2009.