International Journal of Physics and Mathematics
www.physicsjournal.net
Online ISSN: 2664-8644; Print ISSN: 2664-8636; Impact Factor: RJIF 5.46
Received: 27-12-2020; Accepted: 13-01-2021; Published: 04-02-2021
Volume 3; Issue 1; 2021; Page No. 01-05

Application of fixed point theory in D-metric spaces with using banach contraction principal

Mohsin Ahmad Dar[1], Bhawna Agrawal[2]
[1] M. Phil Research Scholar, Department of Mathematics Rabindranath Tagore University Raisen, Madhya Pradesh, India
[2] Associate Professor, Department of Mathematics Rabindranath Tagore University Raisen, Madhya Pradesh, India

Abstract
The D-metric space was introduced by Dhage. In this paper we will use the definition of D-metric space and the concept of Banach contraction principal. Our objective is to find out an application in d-metric space and also discuss about fixed point theory. Mainly we recognized some fixed point theorems in D −metric space, which generalized many results of best mathematicians.

Keywords: D-metric space, banach contraction principal, complete metric space fixed point theory

Introduction

Definition: Let X be a nonempty set, and let $\mathbb{R}$ denote the real numbers. A function $D: X \times X \times X \rightarrow \mathbb{R}$ satisfying the following axioms:

$(D1)$ $D(x,y,z) \geq 0$ $for\ all\ x,y,z \in X$, (Non-Negative)

$(D2)$ $D(x,y,z) = 0\ if\ and\ only\ if\ x = y = z\ (coincide)$

$(D3)$ $D(x,y,z) = D(x,y,z) = \cdots\ (symmetry\ in\ all\ three\ variables),$

$(D4)$ $D(x,y,z) \leq D(x,y,a) + D(x,a,z) + D(a,y,z)\ for\ all\ x,y,z,a \in X.(rectangular\ inequality)$

Is called a generalized metric, or a D-metric on X. The set X together with generalized metric, D is called generalized metric space, or D-metric space, and denoted by (X, D).

An additional property sometimes imposed on a D-metric space (see [3] is,

$(D5)$ $D(x,y,y) \leq D(x,z,z) + D(z,y,y)\ for\ all\ x,y,z \in X$

If $D(x,x,y) = D(x,y,y)\ for\ all\ x,y \in X\ then\ D$ is referred to as a symmetric $D - metric.$

Example 1: (Dhage 1994) for x, y, z in $\mathbb{R}$ define

$D_1(x,y,z) = |x-y| + |y-z| + |z-x|$

$D_\infty(x,y,z) = \max\{|x-y|, |y-z|, |z-x|\}$ Then $(\mathbb{R}, D_1)$ and $(\mathbb{R}, D_\infty)$ are D-metric spaces,

Example 2: (Dhage 1994) Define a function on $X \times X \times X$ by

$$D(x,y,z) = \begin{cases} 0\ if\ x = y = z \\ 1\ otherwise \end{cases}$$

Then D is a D- metric on X and is called the discreate D-metric on X.
Remark: The D-metric given in example 1 and 2 satisfying the following properties:

$(D6)$ $D(x,y,y) \leq D(x,z,z) + D(z,y,y)\ for\ all\ x,y,z \in X$
$(D7)$ $D(x,x,y) = D(x,y,y)$
$(D8)$ $D(x,y,y) \leq D(x,y,z)$
$(D9)$ $D(x,y,z) \leq D(x,y,a) + D(x,a,z) + D(a,y,z)\ for\ all\ x,y,z,a \in X.(rectangular\ inequality)$

Introduction

The Banach principal is a very popular tool in solving existence problems in many branches of mathematics, such as non-linear functional analysis, operator theory, mathematical analysis and general analysis. Fixed point theory is an important branch of functional analysis and topology. In mathematics the Banach contraction principal [1], also known as the Banach fixed point theorem, is one of the main pillars of the theory of metric fixed points. According to the Banach contraction principle, if T is a contraction on a Banach space X, then T has a unique fixed point in X. In general a point which coincides with its transformation is called a fixed point; that is a point $x\ in\ X$ is called fixed point for a self-map T defined on $x\ if\ Tx = x$. Historically many researchers investigated the Banach fixed point theorems in his own ways and presented generalizations, extensions, and application of their findings. Among them in 1992 Dhage introduced the distinguished notation of D-metric spaces called generalized metric space and proved some fixed point theorems.

In 1992 B. C. Dhage [2] projected the concept of a D {metric space in an attempt to attain similar results to those for metric spaces, but in a more general setting. In a following series of papers (including: [3, 4, 5, 6] Dhage presented topological structures in such spaces together with several fixed-point results. These works have been the basis for a considerable number of results by other authors. The idea is clearly generalised of the metric space as above mentioned definition.

The well-known Banach [7] contraction principal states that "If X is a complete metric space and f is a contraction mapping on X into itself, then f has a unique fixed point In X". Numerous mathematicians worked on this principle. Kanan [10] proved that "if T is a Self Mapping of a complete metric space X into itself satisfying:

$d(Tx,Ty)\ \le [d(Tx,x)+d(Ty,y)]$ For all $x,y\ \epsilon\ X$, where $\alpha\ \epsilon\ \left[0,\frac{1}{2}\right]$.

Then T has a unique fixed point in x.

Fisher [3] proved the result with

$d(Tx,Ty)\le[d(Ty,x)+d(Tx,y)]$, for all $x,y\ \in X$, where $\alpha\ \epsilon\ \left[0,\frac{1}{2}\right]$.

A similar conclusion was also obtained by Chaterjee [8].

In 1977, the mathematician Jaggi [12] introduced the rational expression first time as:

$$d(Tx,Ty)\le \alpha d(x,y)+\beta\frac{d(x,Tx)d(y,Ty)}{d(x,y)}$$

for all $x,y\ \epsilon\ X, x\neq y, 0\le\alpha+\beta<1$.

Now,

$$d(Tx,Ty,Tz)\le \alpha d(x,y,z)+\beta\frac{d(x,Tx)d(y,Ty)d(z,Tz)}{d(x,y,z)}$$

For all $x,y,z\ \epsilon\ X, x\neq y\neq z, 0\le\alpha+\beta<1$.

Main Result

We use the concept of Banch contraction principal by the use of $D-$metric space. Let T be a continuous self-map, defined on a complete metric space x. Further T satisfies the following conditions:

$$\begin{aligned}d(Tx,Ty,Tz)\ \le\ &\alpha\frac{d(x,Tx)d(y,Ty)d(z,Tz)+d(x,Tz)d(y,Tx)d(z,Ty)}{d(x,y,z)}+\beta\frac{d(x,Tz)[d(x,Tx)+d(y,Ty)+d(z,Tz)}{d(x,y,z)+d(z,Tz)+d(z,Tx)+d(z,Ty)}\\&+\gamma\frac{d(x,Tx)d(y,Tx)d(z,Tx)+d(y,Ty)d(x,Ty)d(z,Ty)+d(z,Tz)d(x,Tz)d(y,Tz)}{d(x,Tx)+d(y,Tx)+d(z,Tx)+d(y,Ty)+d(x,Ty)+d(z,Ty)+d(z,Tz)+d(x,Tz)+d(y,Tz)}\end{aligned}$$

$$+\delta[d(x,Tx)+d(y,Ty)+d(z,Tz)+\eta[d(x,Tz)+d(y,Tx)+d(z,Ty)]+\mu d(x,y,z).$$

For all $x,y,z\ \epsilon\ X, x\neq y\neq z$ and for $\alpha,\beta,\gamma,\delta,\eta,\mu\ \in[0,1)$, and $2\alpha+\gamma+4\delta+4\eta+2\mu<2, then\ T\ has\ a\ unique\ fixed\ point\ in\ T$

Proof: Let x_0 be an arbitrary point in X, and we define a sequence $\{x_n\}$ by means of iterates of T. By setting, $T^n x_0 = x_n$, where n is a positive integer. If $x_n = x_{n+1}$, for Some n, then we have $T^n x_n = x_n$, then x_n is a fixed point T taking $x_n\neq x_{n+1}$ for all n.

Now $d(x_{n+2},x_{n+1},x_n)=d(Tx_n,Tx_{n-1},Tx_{n-2})$

$$d(Tx_n,Tx_{n-1},Tx_{n-2})\le\alpha\frac{d(x_{n,Tx_n})d(x_{n-1},Tx_{n-1})d(x_{n-2,}Tx_{n-2})+d(x_n,Tx_{n-2})d(x_{n-2,}Tx_{n-1})d(x_{n-1,}Tx_n)}{d(x_{n,}x_{n-1,}x_{n-2})}$$

$$+\beta \frac{d(x_{n-2_n,Tx_n})[d(x_{n-1},Tx_{n-1_n})+d(x_{n-2,}Tx_{n-2_n})+d(x_n,Tx_n)]}{d(x_{n,}x_{n-2_n})+d(x_{n-1},Tx_{n-1_n})+d(x_{n-2,}Tx_{n-2_n})+d(x_n,Tx_n)}$$

$$+\gamma \frac{d(x_{n,Tx_n})d(x_{n-1},Tx_{n-1})d(x_{n-2,}Tx_{n-2})+d(x_{n-1},Tx_{n-1})d(x_{n,,}Tx_{n-1})d(x_{n-2`,}Tx_{n-1})}{d(x_{n,Tx_n})+d(x_{n-1},Tx_{n-1})+d(x_{n-2,}Tx_{n-2})+d(x_{n-1},Tx_{n-1})+d(x_{n,,}Tx_{n-1}) \; d(x_{n-2`,}Tx_{n-1})}$$

$$+\delta[d(x_n,T_{x_n})+d(x_{n-1},T_{x_{n-1}})+d(x_{n-2},T_{x_{n-2}})]+\eta[d(x_{n-1},T_{x_n})+d(x_n,Tx_{n-1})+d(x_{n-2,}Tx_{n-1})]+\mu d(Tx_n,Tx_{n-1},Tx_{n-2})$$
$$\leq \left(\alpha+\frac{\gamma}{3}+\delta+\eta\right)d(x_n,x_{n+1},x_{n+2})+(\delta+\eta+\mu)d(x_{n-2},x_{n-1},x_n)$$

i.e.

$$d(x_{n+2},x_{n+1},x_n) \leq \frac{\delta+\eta+\mu}{1-(\alpha+\frac{\gamma}{3}+\delta+\eta)}d(x_{n-2},x_{n-1},x_n)$$

On applying the same process, we get

$$d(x_{n+2},x_{n+1},x_n) \leq \frac{\delta+\eta+\mu}{1-(\alpha+\frac{\gamma}{3}+\delta+\eta)}d(x_0,x_1,x_2)$$

By the rectangular inequality of $d-metric$ space, we have for $m>n>p$,

$$d(x_n,x_m\, x_p) \leq d(x_n,x_{n+1},x_{n+2})+d(x'_{n+2},x_{n+2},x_{n+3})+d(x'_{n+3},x_{n+4},x_{n+5})+\cdots+d(x'_{p-2}x_{p-1}x_p)$$
$$\leq (s^n+s^{n+1}+s^{n+2}+\cdots|s^{p-1})d(x_0,T_{x_0})$$

Where s $=\frac{\delta+\eta+\mu+\mu}{1-(\alpha+\frac{\gamma}{3}+\delta+\eta)}<1$.

Therefore,

$$d(x_n,x_m\, x_p) \leq \frac{s^n}{1-s}d(x_0,T_{x_0}) \to 0 \; as\; m,n,p \to \infty.$$

So, x_n is a Cauchy sequence in x, so by the completeness of X, there is a point $u \in X$ such that $x_n \to u \; as \; n \to \infty$. Further, the continuity of T in X implies

$$T(u)=T\left(\lim_{n\to\infty} x_n\right)=\lim_{n\to\infty} Tx_n = \lim n \to \infty x_{n+1}=u.$$

Therefore, u is a fixed point of T in x.

Uniqueness

Suppose if there is any other $v \neq u \neq w$ in X such that $T(v)=v$, then $d(u,v,w)=d(T_u,T_v,T_w)$

$$d(u,v,w) \leq \alpha \frac{d(u,Tu)d(v,Tv)d(w,Tw)+d(u,Tv)d(v,Tw)d(w,Tu)}{d(u,v,w)}$$

$$+\beta \frac{d(u,Tw)[d(u,Tu)+d(v,Tv)+d(w,Tw)]}{d(u,v,w)+d(u,v,Tw)+d(u,w,Tv)+d(v,w,Tu)}$$

$$+\gamma \frac{d(u,Tu)d(v,Tu)d(w,Tu)+d(w,Tv)d(v,Tv)d(u,Tv)+d(w,Tw)d(u,Tw)d(v,Tw)}{d(u,Tu)d(v,Tu)d(w,Tu)d(w,Tv)d(v,Tv)d(u,Tv)d(w,Tw)d(u,Tw)d(v,Tw)}$$
$$+\delta[d(u,Tu)+d(v,Tv)+d(w,Tw)]+\eta[d(u,Tw)+d(v,Tu)+d(w,Tv)]$$
$$+\mu d(u,v,w)$$

i.e.

$$d(u,v,w) \leq [\alpha+\beta+3\eta+\mu]d(u,v,w).$$

This is a contradiction because, $2\alpha+\gamma+4\delta+4\eta+2\mu<2$. Hence u is the unique fixed point.

Now we are going to prove a fascinating result in which T is not necessarily continuous in X, but T^p is continuous for some positive integer P, T^p is continuous, then T has a unique fixed point.

Theorem

Let T be a continuous self map, defined on a complete metric space (X, d), such that for some positive integer m, T satisfies the following conditions:

$$d(T^m x, T^m y, T^m z) \le \alpha \frac{d(x,T^m x)d(y,T^m y)d(z,T^m z) + d(x,T^m z)d(y,T^m x)d(z,T^m y)}{d(x,y,z)}$$

$$+\beta \frac{d(x,T^m z)[d(x,T^m x) + d(y,T^m y) + d(z,T^m z)]}{d(x,y,z) + d(x,y,T^m z) + d(y,z,T^m x) + d(z,x,T^m y)}$$

$$+\gamma \frac{d(x,T^m x)d(z,T^m x) + d(y,T^m y)d(x,T^m y) + d(x,T^m z)d(z,T^m z)}{d(x,T^m x) + d(z,T^m x) + d(y,T^m y) + d(x,T^m y) + d(x,T^m z) + d(z,T^m z)} + \delta[d(x,T^m x) + d(y,T^m y) + d(z,T^m z)]$$
$$+ \eta d(x,y,T^m z) + d(y,z,T^m x) + d(z,x,T^m y)] + \mu d(x,y,z)$$

For all $x, y, z \in X, x \neq y \neq z$ and $\alpha, \beta, \gamma, \delta, \eta, \mu \ge 0$, with $2\alpha + \gamma + 4\delta + 4\eta + 2\mu < 2$.

If T^m is continuous then T has unique fixed-point.

Proof

By the previous theorem already proved that, Let T be a self map defined on a complete metric space (X, d) such that $(D4)$ holds. If some positive integer $P, T^p$is continuous, then T has a unique fixed point. We assume that T^m has a unique fixed point. Also
$Tu = T(T^m u) = T^m(Tu)$.
Which implies $Tu = u$, further since fixed point of T is a fixed point of T^m, and T^m has a unique fixed point u, it follows that u is the unique fixed point of T.

Example

Let $X = [0,1]$ with the usual metric and $T: X \to X$ be defined by
$Tx = \{0, \text{When } 0 \le x \le \frac{1}{2}$
$= \{\frac{1}{2}, \text{When} \frac{1}{2} < x \le 1.$

It is clear that T is discontinuous and does not satisfy $(D4)$ for any $\alpha, \beta, \gamma, \delta, \eta, \mu \in [0,1)$ with $2\alpha + \gamma + 4\delta + 4\eta + 2\mu < 2$. When $x = \frac{1}{2}, y = 1$. But it can be easily see that T^2 is continuous and satisfies the condition of theorem 2, and 0 is unique fixed point of T^2 and also of T.

Remark

1. If $\alpha = \beta = \gamma = \delta = \eta = 0$, then theorem 1 reduce to Banach [1].
2. If $\alpha = \beta = \gamma = \delta = \mu = 0$, then theorem 1 reduce to Kannan [4].
3. If $\alpha = \beta = \gamma = \eta = 0$, then theorem 1 reduce to Chatterjee [2].
4. If $\alpha = \beta = \gamma = \delta = 0$, then theorem 1 reduce to Fisher [3].
5. If $\alpha = \beta = \gamma = 0$, then theorem 1 reduce to Reich [5].

References

1. Ramabhadrasarma, Sambasivaro S. On D-metric spaces. Journal of Global Research in Mathematical Archives, 201, 1(12).
2. Dhage BC. Generalized Metric Space and Mapping With Fixed Point, Bull. Cal. Math. Soc. 1992; 84:329-336.
3. Dhage BC. Generalized Metric Space and Topological Structure I, An.stiint.Univ. Al.I. Cuza Ia_si. Mat (N.S). 2000; 46:3-24.
4. Dhage BC. On Generalized Metric Spaces and Topological Structure II, Pure. Appl. Math. Sci. 1994; 40:37-41.
5. Dhage BC. On Continuity of Mappings in D-metric Spaces, Bull. Cal. Math. Soc. 1994; 86:503-508.
6. Dhage BC. Generalized D {Metric Spaces and Multi {valued Contraction Mappings, An. Stint. Univ Al.l. cuza lasi. Mat (N.S). 1998; 44:179-200.
7. Banach S. Surles operation danes less ensembles abstracts application aux equation integrals. Fun. Math. 1992; 3:133-181.
8. Chatterjee SK. Fixed point theorems. Comptes. Rend. Acad. Bulgaria Sci. 1972; 25:727-730.
9. Fisher B. A fixed point theorem for compact metric space. Publ. Mat Inst. Math. 1976; 25:193-194.
10. Kannan R. Some results on fixed points-II. Buletin of Calcutta Math. Soc. 1969; 60:71-76.
11. Reich S. Some remarks concerning contraction mappings. Canada Math. Bull. 1971; 1:121-124.

12. Jaggi DS. Some unique fixed theorems. Indian journal of pure and Applied Mathematics. 1977; 8:223-230.
13. Jaggi DS, Dass BK. An extension of Banach contraction theorem through rational expression. Bull. Cal. Math, 1980, 261-266.