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Parameter estimation of inverted exponential distribution via Bayesian approach

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Abstract
In this paper, the inverted exponential distribution is considered for Bayesian analysis. The expressions for Bayes estimators of the parameter have been derived under squared error, precautionary, entropy, K-loss, and Al-Bayyati's loss functions by using quasi and gamma priors.

Keywords: Inverted exponential distribution, Bayesian method, quasi and gamma priors, squared error, precautionary, entropy, K-loss, and Al-Bayyati's loss functions.

Introduction

The inverted exponential distribution is studied as a prospective life distribution (Sanku Dey [1]). The probability density function of inverted exponential distribution is given by

$$f\left(x;\theta\right)=\theta x^{-2}e^{-\theta/x} \quad ; x\geq 0,\ \theta>0. \tag{1}$$

The joint density function or likelihood function of (1) is given by

$$f\left(\underline{x};\theta\right)=\theta^{n}\left(\prod_{i=1}^{n}x_i^{-2}\right)e^{-\theta\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)}. \tag{2}$$

The log likelihood function is given by

$$log\ f\left(\underline{x};\theta\right)=n\,log\,\theta+log\left(\prod_{i=1}^{n}x_i^{-2}\right)-\theta\sum_{i=1}^{n}\left(\frac{1}{x_i}\right) \tag{3}$$

Differentiating (3) with respect to θ and equating to zero, we get the maximum likelihood estimator of θ which is given as

$$\hat{\theta}=\frac{n}{\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)}. \tag{4}$$

2. Bayesian Method of Estimation

The Bayesian inference procedures have been developed generally under squared error loss function

$$L\left(\hat{\theta},\theta\right)=\left(\hat{\theta}-\theta\right)^{2}. \tag{5}$$

The Bayes estimator under the above loss function, say, $\hat{\theta}_s$ is the posterior mean, i.e,

$$\hat{\theta}_S=E\left(\theta\right). \tag{6}$$

Zellner [2], Basu and Ebrahimi [3] have recognized that the inappropriateness of using symmetric loss function. Norstrom [4] introduced precautionary loss function is given as

$$L\left(\hat{\theta},\theta\right)=\frac{\left(\hat{\theta}-\theta\right)^2}{\hat{\theta}}. \tag{7}$$

The Bayes estimator under precautionary loss function is denoted by $\hat{\theta}_p$ and is obtained by solving the following equation

$$\hat{\theta}_P=\left[E\left(\theta^2\right)\right]^{1/2}. \tag{8}$$

In many practical situations, it appears to be more realistic to express the loss in terms of the ratio $\frac{\hat{\theta}}{\theta}$. In this case, Calabria and Pulcini [5] points out that a useful asymmetric loss function is the entropy loss

$$L(\delta)\propto\left[\delta^p-p\ \log_e(\delta)-1\right]$$

where $\delta=\frac{\hat{\theta}}{\theta}$, and whose minimum occurs at $\hat{\theta}=\theta$. Also, the loss function $L(\delta)$ has been used in Dey *et al.* [6] and Dey and Liu [7], in the original form having $p=1$. Thus $L(\delta)$ can written be as

$$L(\delta)=b\left[\delta-\log_e(\delta)-1\right];\ b>0. \tag{9}$$

The Bayes estimator under entropy loss function is denoted by $\hat{\theta}_E$ and is obtained by solving the following equation

$$\hat{\theta}_E=\left[E\left(\frac{1}{\theta}\right)\right]^{-1}. \tag{10}$$

Wasan [8] proposed the K-loss function which is given as

$$L\left(\hat{\theta},\theta\right)=\frac{\left(\hat{\theta}-\theta\right)^2}{\hat{\theta}\theta}. \tag{11}$$

Under K-loss function the Bayes estimator of θ is denoted by $\hat{\theta}_K$ and is obtained as

$$\hat{\theta}_K=\left[\frac{E(\theta)}{E(1/\theta)}\right]^{\frac{1}{2}}. \tag{12}$$

Al-Bayyati [9] introduced a new loss function using Weibull distribution which is given as

$$L\left(\hat{\theta},\theta\right)=\theta^c\left(\hat{\theta}-\theta\right)^2. \tag{13}$$

Under Al-Bayyati's loss function the Bayes estimator of θ is denoted by $\hat{\theta}_{Al}$ and is obtained as

$$\hat{\theta}_{Al} = \frac{E\left(\theta^{c+1}\right)}{E\left(\theta^{c}\right)}. \tag{14}$$

Let us consider two prior distributions of θ to obtain the Bayes estimators.
(i) Quasi-prior: For the situation where we have no prior information about the parameter θ, we may use the quasi density as given by

$$g_1(\theta) = \frac{1}{\theta^d} \;;\; \theta > 0,\; d \geq 0, \tag{15}$$

where $d = 0$ leads to a diffuse prior and $d = 1$, a non-informative prior.
(ii) Gamma prior: Generally, the gamma density is used as prior distribution of the parameter θ given by

$$g_2(\theta) = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta} \;;\; \theta > 0. \tag{16}$$

3. Posterior density under $g_1(\theta)$

The posterior density of θ under $g_1(\theta)$, on using (2), is given by

$$f(\theta/\underline{x}) = \frac{\theta^n \left(\prod_{i=1}^{n} x_i^{-2}\right) e^{-\theta\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)} \theta^{-d}}{\int_0^{\infty} \theta^n \left(\prod_{i=1}^{n} x_i^{-2}\right) e^{-\theta\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)} \theta^{-d} d\theta}$$

$$= \frac{\theta^{n-d}\; e^{-\theta\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)}}{\int_0^{\infty} \theta^{n-d}\; e^{-\theta\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)} d\theta}$$

$$= \frac{\left(\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)\right)^{n-d+1}}{\Gamma(n-d+1)} \theta^{n-d}\; e^{-\theta\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)} \tag{17}$$

Theorem 1. On using (17), we have

$$E\left(\theta^{c}\right) = \frac{\Gamma(n-d+c+1)}{\Gamma(n-d+1)} \left(\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)\right)^{-c}. \tag{18}$$

Proof. By definition,

$$E\left(\theta^{c}\right) = \int \theta^{c} f(\theta/\underline{x}) d\theta$$

$$= \frac{\left(\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)\right)^{n+d-1}}{\Gamma(n+d-1)} \int_0^\infty \theta^{-(n-d+c)} \ e^{-\theta\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)} d\theta$$

$$= \frac{\left(\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)\right)^{n-d+1}}{\Gamma(n-d+1)} \frac{\Gamma(n-d+c+1)}{\left(\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)\right)^{n-d+c+1}}$$

$$= \frac{\Gamma(n-d+c+1)}{\Gamma(n-d+1)}\left(\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)\right)^{-c} .$$

From equation (18), for $c=1$, we have

$$E(\theta) = (n-d+1)\left[\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)\right]^{-1} . \tag{19}$$

From equation (18), for $c=2$, we have

$$E(\theta^2) = \left[(n-d+2)(n-d+1)\right]\left[\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)\right]^{-2} . \tag{20}$$

From equation (18), for $c=-1$, we have

$$E\left(\frac{1}{\theta}\right) = \frac{1}{(n-d)}\sum_{i=1}^{n}\left(\frac{1}{x_i}\right). \tag{21}$$

From equation (18), for $c=c+1$, we have

$$E(\theta^{c+1}) = \frac{\Gamma(n-d+c+2)}{\Gamma(n-d+1)}\left(\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)\right)^{-(c+1)} . \tag{22}$$

4. Bayes Estimators under $g_1(\theta)$

From equation (6), on using (19), the Bayes estimator of θ under squared error loss function is given by

$$\hat{\theta}_S = (n-d+1)\left[\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)\right]^{-1} . \tag{23}$$

From equation (8), on using (20), the Bayes estimator of θ under precautionary loss function is given by

$$\hat{\theta}_P = \left[(n-d+2)(n-d+1)\right]^{\frac{1}{2}} \left(\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)\right)^{-1}. \tag{24}$$

From equation (10), on using (21), the Bayes estimator of θ under entropy loss function is given by

$$\hat{\theta}_E = (n-d)\left(\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)\right)^{-1}. \tag{25}$$

From equation (12), on using (19) and (21), the Bayes estimator of θ under K-loss function is given by

$$\hat{\theta}_K = \left[(n-d+1)(n-d)\right]^{\frac{1}{2}} \left(\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)\right)^{-1}. \tag{26}$$

From equation (14), on using (18) and (22), the Bayes estimator of θ under Al-Bayyati's loss function is given by

$$\hat{\theta}_{Al} = (n-d+c+1)\left(\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)\right)^{-1}. \tag{27}$$

5. Posterior density under $g_2(\theta)$

Under $g_2(\theta)$, the posterior density of θ, using equation (2), is obtained as

$$f(\theta/\underline{x}) = \frac{\theta^n\left(\prod_{i=1}^{n} x_i^{-2}\right) e^{-\theta\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)} \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}}{\int_0^\infty \theta^n\left(\prod_{i=1}^{n} x_i^{-2}\right) e^{-\theta\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)} \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta}d\theta}$$

$$= \frac{\theta^{n+\alpha-1}\, e^{-\left(\beta+\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)\right)\theta}}{\int_0^\infty \theta^{n+\alpha-1}\, e^{-\left(\beta+\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)\right)\theta} d\theta}$$

$$= \frac{\theta^{n+\alpha-1}\, e^{-\left(\beta+\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)\right)\theta}}{\Gamma(n+\alpha)\Big/\left(\beta+\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)\right)^{n+\alpha}}$$

$$= \frac{\left(\beta+\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)\right)^{n+\alpha}}{\Gamma(n+\alpha)}\theta^{n+\alpha-1}\, e^{-\left(\beta+\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)\right)\theta} \tag{28}$$

Theorem 2. On using (28), we have

$$E\left(\theta^{c}\right)=\frac{\Gamma\left(n+\alpha+c\right)}{\Gamma\left(n+\alpha\right)}\left(\beta+\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)\right)^{-c}. \tag{29}$$

Proof. By definition,

$$E\left(\theta^{c}\right)=\int\theta^{c} f\left(\theta/\underline{x}\right)d\theta$$

$$=\frac{\left(\beta+\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)\right)^{n+\alpha}}{\Gamma\left(n+\alpha\right)}\int_0^{\infty}\theta^{n+\alpha+c-1}\, e^{-\left(\beta+\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)\right)\theta}d\theta$$

$$=\frac{\left(\beta+\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)\right)^{n+\alpha}}{\Gamma\left(n+\alpha\right)}\frac{\Gamma\left(n+\alpha+c\right)}{\left(\beta+\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)\right)^{n+\alpha+c}}$$

$$=\frac{\Gamma\left(n+\alpha+c\right)}{\Gamma\left(n+\alpha\right)}\left(\beta+\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)\right)^{-c}.$$

From equation (29), for $c=1$, we have

$$E\left(\theta\right)=\left(n+\alpha\right)\left(\beta+\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)\right)^{-1}. \tag{30}$$

From equation (29), for $c=2$, we have

$$E\left(\theta^{2}\right)=\left[\left(n+\alpha+1\right)\left(n+\alpha\right)\right]\left(\beta+\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)\right)^{-2}. \tag{31}$$

From equation (29), for $c=-1$, we have

$$E\left(\frac{1}{\theta}\right)=\frac{1}{\left(n+\alpha-1\right)}\left(\beta+\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)\right). \tag{32}$$

From equation (29), for $c=c+1$, we have

$$E\left(\theta^{c+1}\right)=\frac{\Gamma\left(n+\alpha+c+1\right)}{\Gamma\left(n+\alpha\right)}\left(\beta+\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)\right)^{-(c+1)}. \tag{33}$$

6. Bayes Estimators under $g_2(\theta)$

From equation (6), on using (30), the Bayes estimator of θ under squared error loss function is given by

$$\hat{\theta}_S = (n+\alpha)\left(\beta + \sum_{i=1}^{n}\left(\frac{1}{x_i}\right)\right)^{-1}. \tag{34}$$

From equation (8), on using (31), the Bayes estimator of θ under precautionary loss function is given by

$$\hat{\theta}_P = \left[(n+\alpha+1)(n+\alpha)\right]^{\frac{1}{2}}\left(\beta + \sum_{i=1}^{n}\left(\frac{1}{x_i}\right)\right)^{-1}. \tag{35}$$

From equation (10), on using (32), the Bayes estimator of θ under entropy loss function is given by

$$\hat{\theta}_E = (n+\alpha+1)\left(\beta + \sum_{i=1}^{n}\left(\frac{1}{x_i}\right)\right)^{-1}. \tag{36}$$

From equation (12), on using (30) and (32), the Bayes estimator of θ under K-loss function is given by

$$\hat{\theta}_K = \left[(n+\alpha)(n+\alpha-1)\right]^{\frac{1}{2}}\left(\beta + \sum_{i=1}^{n}\left(\frac{1}{x_i}\right)\right)^{-1}. \tag{37}$$

From equation (14), on using (29) and (33), the Bayes estimator of θ under Al-Bayyati's loss function is given by

$$\hat{\theta}_{Al} = (n+\alpha+c)\left(\beta + \sum_{i=1}^{n}\left(\frac{1}{x_i}\right)\right)^{-1}. \tag{38}$$

Conclusion

In this paper, we have obtained a number of estimators of parameter of inverted exponential distribution. In equation (4) we have obtained the maximum likelihood estimator of the parameter. In equation (23), (24), (25), (26) and (27) we have obtained the Bayes estimators under different loss functions using quasi prior. In equation (34), (35), (36), (37) and (38) we have obtained the Bayes estimators under different loss functions using gamma prior. In the above equation, it is clear that the Bayes estimators depend upon the parameters of the prior distribution.

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