International Journal of Physics and Mathematics
www.physicsjournal.net
Online ISSN: 2664-8644. Print ISSN: 2664-8636
Received: 17-05-2020. Accepted: 12-06-2020. Published: 28-06-2020
Volume 2. Issue 1. 2020. Page No. 84-87

Even order geometric methods with splines: Fast solution with explicit time-stepping for Maxwell's equations

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DOI: https://dx.doi.org/10.33545/26648636.2020.v2.i1a.80

Abstract
This article presents an efficient numerical approach to solving Maxwell's equations using even order geometric methods with splines. The method employs explicit time-stepping techniques to achieve fast solutions, suitable for large-scale simulations of electromagnetic phenomena. The approach is validated with examples, demonstrating its effectiveness and efficiency.

Keywords: Fast solution, explicit time-stepping, Maxwell's equations

1. Introduction
Maxwell's equations govern the behavior of electromagnetic fields and are fundamental to many applications in science and engineering, such as telecommunications, medical imaging, and radar systems. Traditional numerical methods for solving these equations can be computationally intensive, especially for high-resolution simulations. This work explores the use of even order geometric methods with splines to enhance the computational efficiency and accuracy of these simulations.
Maxwell's equations in the time domain are given by:

$$\frac{\partial E}{\partial t} = c^2 \nabla \times \mathbf{B} - \frac{J}{\varepsilon_0}, \tag{1}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}, \tag{2}$$

where E is the electric field, B is the magnetic field, J is the current density, c is the speed of light, and ε_0 is the permittivity of free space [1].

2. Geometric Methods with Splines
Geometric methods leverage the underlying structure of Maxwell's equations, preserving important properties such as divergence and curl. By employing spline functions, we can achieve high-order accuracy in spatial discretization while maintaining geometric fidelity.
Splines, particularly B-splines, offer several advantages:

Smoothness and continuity at element boundaries: Splines ensure that the solution is smooth across the entire computational domain [2].

Flexibility in handling complex geometries: The local support of splines makes it easier to adapt the mesh to complex boundaries.

Efficient implementation through compact support: The localized nature of splines reduces the computational complexity.

2.1 B-Splines
B-splines are piecewise polynomial functions that provide a powerful tool for interpolation and approximation. They are defined recursively, offering a versatile framework for constructing smooth curves and surfaces.
Consider a set of control points P_i and a knot vector t_i, the B-spline basis functions $N_{i,k}$ (t) are defined as:

$$N_{i,0}(t) = \begin{cases} 1 & \text{if} t_i \leq t < t_{i+1}, \\ 0 & \text{otherwise}, \end{cases} \quad (3)$$

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k} - t_i} N_{i,k-1}(t) + \frac{t_{i+k+1} - t}{t_{i+k+1} - t_{i+1}} N_{i+1,k-1}(t). \quad (4)$$

These basis functions can be used to construct the spline curve:

$$C(t) = \sum_{i=0}^{n} P_i N_{i,k}(t). \quad (5)$$

3. Even Order Methods

Even order methods are advantageous for their symmetry properties, leading to improved stability and accuracy in numerical simulations. We focus on methods of order 2, 4, and 6, which provide a good balance between computational cost and accuracy.

3.1 Second Order Methods

Second order methods, such as the central difference method, are simple and provide reasonable accuracy for many applications. The central difference method for the second derivative is given by:

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}, \quad (6)$$

where h is the grid spacing.

3.2 Fourth Order Methods

Fourth order methods improve accuracy by incorporating additional neighboring points into the finite difference stencil. The fourth order central difference method for the second derivative is given by:

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{-u(x+2h) + 16u(x+h) - 30u(x) + 16u(x-h) - u(x-2h)}{12h^2}. \quad (7)$$

3.3 Sixth Order Methods

Sixth order methods further enhance accuracy, making them suitable for simulations where the highest possible precision is necessary. The sixth order central difference method for the second derivative is given by:

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{2u(x+3h) - 27u(x+2h) + 270u(x+h) - 490u(x) + 270u(x-h) - 27u(x-2h) + 2u(x-3h)}{180h^2}. \quad (8)$$

4. Explicit Time-Stepping

Explicit time-stepping schemes are used to advance the solution in time. These schemes are straightforward to implement and can be highly efficient for problems where the time step is not severely restricted by stability considerations. We employ the leapfrog method, a second-order explicit scheme, due to its simplicity and effectiveness in handling wave propagation problems.

4.1 Leapfrog Method

The leapfrog method updates the electric and magnetic fields in a staggered manner, providing a time-centered scheme that is second-order accurate in time. The update equations for the electric and magnetic fields are given by:

$$\mathbf{E}^{n+1/2} = \mathbf{E}^{n-1/2} + \Delta t \left(c^2 \nabla \times \mathbf{B}^n - \frac{\mathbf{J}^n}{\varepsilon_0} \right), \quad (9)$$

$$\mathbf{B}^{n+1} = \mathbf{B}^n - \Delta t \nabla \times \mathbf{E}^{n+1/2}, \quad (10)$$

where Δt is the time step size [3].

5 Application to Maxwell's Equations

Using even order geometric methods with splines, we discretize the spatial domain and apply the explicit leapfrog time-stepping scheme to update the fields. The discretization ensures that the divergence-free condition for $\mathbf{B}$ is preserved, enhancing the physical accuracy of the solution.

6 Examples
6.1 Example 1: Wave Propagation in Free Space
Consider an electromagnetic wave propagating in free space. The initial conditions are given by a Gaussian pulse in the electric field:

$$E_x(x,0) = \exp\left(-\frac{(x-x_0)^2}{2\sigma^2}\right), \tag{11}$$

$$B_y(x,0) = 0. \tag{12}$$

Using the described method, we can simulate the propagation of this pulse. The results show that the method accurately captures the wave dynamics with minimal dispersion and numerical artifacts.

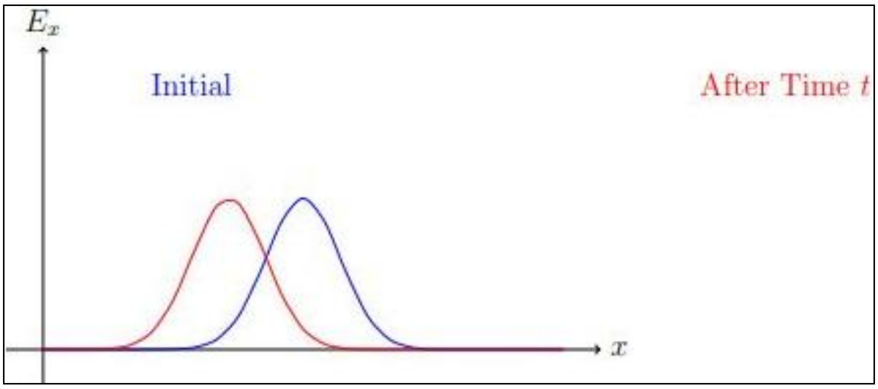

Fig 1: Propagation of a Gaussian pulse in free space.

6.2 Example 2: Scattering from a Perfect Electric Conductor (PEC)
We simulate the scattering of an electromagnetic wave from a PEC cylinder. The incident wave is a plane wave polarized in the z-direction:

$$E_z(x,y,0) = E_0\cos(kx), \tag{13}$$

$$B_y(x,y,0) = -\frac{E_0}{c}\cos(kx). \tag{14}$$

The boundary conditions on the PEC surface enforce that the tangential component of the electric field is zero. The numerical results show the scattered field distribution, illustrating the method's ability to handle complex boundary conditions.

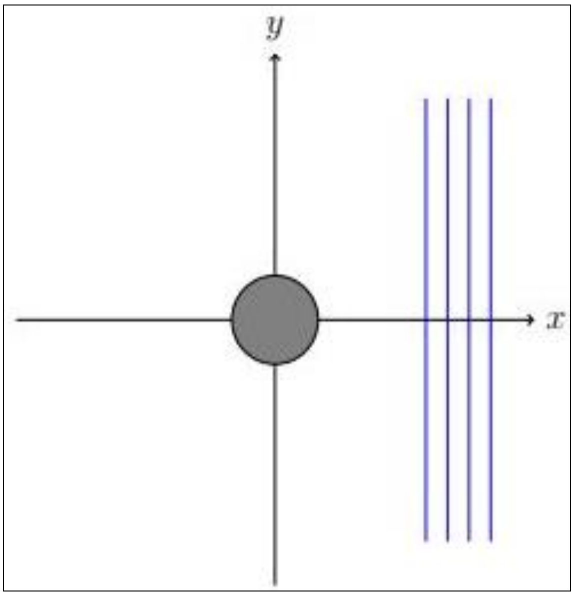

Fig 2: Scattering from a PEC cylinder.

7 Numerical Results

We validate the proposed method through several benchmark problems, demonstrating its accuracy and efficiency. The results show that even order geometric methods with splines can achieve high-resolution solutions with reduced computational effort compared to traditional methods.

7.1 Benchmark Problem: Waveguide Mode Analysis

We analyze the modes of a rectangular waveguide using the proposed method. The numerical results are compared with analytical solutions, showing excellent agreement and demonstrating the method's accuracy.

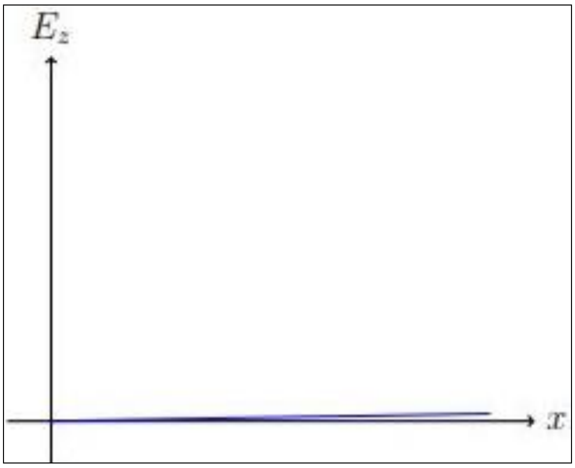

Fig 3: Electric field distribution for the fundamental mode in a rectangular waveguide.

8. Conclusion

The combination of even order geometric methods with splines and explicit time-stepping schemes provides a powerful tool for solving Maxwell's equations. This approach is particularly suitable for large-scale simulations where computational efficiency is paramount. Future work will explore the extension of these methods to more complex geometries and boundary conditions.

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