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A class of fifth order block hybrid Adams Moulton's method for solving stiff initial value problems

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Abstract
This paper is concern with the derivation of the continuous formulations of hybrid Adams-Moulton Method for step number $k = 1, 2$ and 3 incorporating three, two and one off-grid points respectively by multistep collocation method using matrix inversion technique. The discrete schemes used in block form were derived from their respective continuous formulations. The convergence analysis were carried and found that the methods are convergent. The region of absolute stability of the methods were plotted in a block form and they are all $A(\alpha)$ - stable. Some linear stiff problems were solved using the schemes. It was observed that the schemes for step number $k = 3$ incorporating one off-grid point perform better than the schemes for step number $k = 2$ & 1 incorporating two and three off-grid points respectively.

Keywords: fifth order, hybrid, stiff problems, off-grid point

1. Introduction

The numerical methods that have been developed for solving initial value problems are one step method because they only use information about the solution at time t_n to approximate the solution at time t_{n+1}. As n increases, that means there are additional values of solution at previous times that could be helpful, but are unused.

The linear multistep method (LMM) is time-stepping method that does use this information and has a general form as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \tag{1}$$

Where k is the number of step in the method and h is the step size. By convention $\alpha_0 = 1$ so that y_{n+1} can be conventionally expressed in terms of other values if $\beta_0 = 0$, the LMM is said to be explicit otherwise is said to implicit.

The Adams-Moulton method (AMM) is a family of implicit LMM for the numerical integration of ordinary differential equations and is especially used for the solution of stiff differential equations. A popular example of this method is trapezoidal rule, i.e. $y_{n+1} - y_n = \frac{h}{2}\left(f_{n+1} + f_n\right)$

The Adams-Moulton methods are solely due to John Couch Adams. The name of Forest Ray Moulton became associated with these methods as in [1], because he realized that they could be used in tandem with the Adams-Bash forth method as a predictor-corrector pair. The ADMs can also take the forms (1.1) which have the first characteristics equation of the form

$$\rho\left(\xi\right) = \xi^{k} - \xi^{k-1} \tag{2}$$

Although, most of LMMs including ADMs have problems of poor stability property as the step number increases. However, these drawbacks can be lessened by reducing the step number without reduction in its order, while at the same time satisfying the condition of zero stability.

This aims was achieved in the mid 1960's with the introduction of modified LMMs by breeding the LMMs and Runge-Kutta Methods. The modified LMMs are also called Hybrid methods because they incorporate function evaluation at a point between grid points called off-grid point.

The general form of modified LMMs is given by

$$\sum_{j=0}^{k}\alpha_j y_{n+j} = h\sum_{j=0}^{k}\beta_j f_{n+j} + h\beta_r f_{n+r} \qquad (3)$$

Where $j=\{0,1,2,\ldots,k\}$ and $r=\frac{m}{n}$ with $m\geq 1$, $n>1$ and $m\neq n$.

The hybrid schemes have been developed since the 1960's and started by [2, 3, 4, 5], but these methods have not yet received a great deal of attention. [6, 7, 8, 9, 10] have all converted hybrids ones into continuous forms through the idea of multistep collocation. The continuous multistep methods produces piece-wise polynomial solution over k-step for the first order differential system.

2. Derivation Techniques

2.1 Derivation of Multistep Collocation Method

In [9] a *k*-step multistep collocation method with *m* collocation points was obtained as

$$y(x) = \sum_{j=0}^{t-1}\alpha_j(x)y_{n+j} + h\sum_{j=0}^{m-1}\beta_j(x)f(x, y(x)) \qquad (4)$$

Where $\alpha_j(x)$ and $\beta_j(x)$ are continuous coefficients of the method defined as

$$\alpha_j(x) = \sum_{i=0}^{t+m-1}\alpha_{j,i+1}x^i \quad \text{For } j=\{0,1,\ldots,t-1\} \qquad (5)$$

$$h\beta_j(x) = \sum_{i=0}^{t+m-1}h\beta_{j,i+1}x^i \quad \text{For } j=\{0,1,\ldots,m-1\} \qquad (6)$$

Where $x_0, \ldots, x_{m-1}$ are the *m* collocation points and x_{n+j}, $j = 0, 1, 2, \ldots, t-1$ are the *t* arbitrarily chosen interpolation points.

To get $\alpha_j(x)$ and $\beta_j(x)$, Sirisena (1997) arrived at a matrix equation of the form

DC = I (7)

Where *I* is the identity matrix of dimension $(t+m)\times(t+m)$ while *D* and *C* are matrices defined as

$$D = \begin{bmatrix} 1 & x_n & x_n^2 & \cdots & x_n^{t+m-1} \\ 1 & x_{n+1} & x_{n+1}^2 & \cdots & x_{n+1}^{t+m-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{n+t-1} & x_{n+t-1}^2 & \cdots & x_{n+t-1}^{t+m-1} \\ 0 & 1 & 2x_0 & \cdots & (t+m-1)x_0^{t+m-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2x_{m-1} & \cdots & (t+m-1)x_{m-1}^{t+m-2} \end{bmatrix} \qquad (8)$$

$$C=\begin{bmatrix} \alpha_{0,1} & \alpha_{1,1} & \cdots & \alpha_{t-1,1} & h\beta_{0,1} & \cdots & h\beta_{m-1,1} \\ \alpha_{0,2} & \alpha_{1,2} & \cdots & \alpha_{t-1,2} & h\beta_{0,2} & \cdots & h\beta_{m-1,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{0,t+m} & \alpha_{1,t+m} & \cdots & \alpha_{t-1,t+m} & h\beta_{0,t+m} & \cdots & h\beta_{m-1,t+m} \end{bmatrix} \tag{9}$$

It follows from (7) that the columns of $C = D^{-1}$ give the continuous coefficients of the continuous scheme (4).

2.2 Derivation of Hybrid Adams Mouton Method (HAMM)

For step number k = 1 incorporating three off-grid collocation points, here the number of interpolation points, t = 1 and the number of collocation points = 5. Therefore, (4) becomes

$$y(x)=\alpha_0(x)y_n+h\beta_0(x)f_n+h\beta_{\frac{1}{4}}(x)f_{n+\frac{1}{4}}+h\beta_{\frac{1}{2}}(x)f_{n+\frac{1}{2}}+h\beta_{\frac{3}{4}}(x)f_{n+\frac{3}{4}}+h\beta_1(x)f_{n+1}$$

The matrix *D*in (8) becomes

$$D=\begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 \\ 0 & 1 & 2x_{n+\frac{1}{4}} & 3x_{n+\frac{1}{4}}^2 & 4x_{n+\frac{1}{4}}^3 & 5x_{n+\frac{1}{4}}^4 \\ 0 & 1 & 2x_{n+\frac{1}{2}} & 3x_{n+\frac{1}{2}}^2 & 4x_{n+\frac{1}{2}}^3 & 5x_{n+\frac{1}{2}}^4 \\ 0 & 1 & 2x_{n+\frac{3}{4}} & 3x_{n+\frac{3}{4}}^2 & 4x_{n+\frac{3}{4}}^3 & 5x_{n+\frac{3}{4}}^4 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 \end{bmatrix}$$

The inverse of the matrix $C = (D^{-1})$ is computed using Maple 17 from which the following continuous scheme is obtained using (4) and evaluating it at $x=x_{n+\frac{1}{4}}$, $x=x_{n+\frac{1}{2}}$, $x=x_{n+\frac{3}{4}}$ and $x=x_{n+1}$, The following discrete schemes are obtained

$$\left.\begin{aligned} y_{n+\frac{1}{4}} &= y_n+\frac{251}{2880}hf_n+\frac{323}{1440}hf_{n+\frac{1}{4}}-\frac{11}{120}hf_{n+\frac{1}{2}}+\frac{53}{1440}hf_{n+\frac{3}{4}}-\frac{19}{2880}hf_{n+1} \\ y_{n+\frac{1}{2}} &= y_n+\frac{29}{360}hf_n+\frac{31}{90}hf_{n+\frac{1}{4}}+\frac{1}{15}hf_{n+\frac{1}{2}}+\frac{1}{90}hf_{n+\frac{3}{4}}-\frac{1}{360}hf_{n+1} \\ y_{n+\frac{3}{4}} &= y_n+\frac{27}{320}hf_n+\frac{51}{160}hf_{n+\frac{1}{4}}+\frac{9}{40}hf_{n+\frac{1}{2}}+\frac{21}{160}hf_{n+\frac{3}{4}}-\frac{3}{320}hf_{n+1} \\ y_{n+1} &= y_n+\frac{7}{9}hf_n+\frac{16}{45}hf_{n+\frac{1}{4}}+\frac{2}{15}hf_{n+\frac{1}{2}}+\frac{16}{45}hf_{n+\frac{3}{4}}+\frac{7}{90}hf_{n+1} \end{aligned}\right\} \tag{10}$$

For Step number k = 2 incorporating two off-grid collocation points, here the number of interpolation points, t = 1 and the number of collocation points, m = 5. Therefore, (4), becomes

$$y(x)=\alpha_1(x)y_{n+1}+h\beta_0(x)f_n+h\beta_{\frac{1}{2}}(x)f_{n+\frac{1}{2}}+h\beta_1(x)f_{n+1}+h\beta_{\frac{3}{2}}(x)f_{n+\frac{3}{2}}+h\beta_2(x)f_{n+2}$$

The matrix *D*in (8) becomes

$$D=\begin{bmatrix} 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 \\ 0 & 1 & 2x_{n+\frac{1}{2}} & 3x_{n+\frac{1}{2}}^2 & 4x_{n+\frac{1}{2}}^3 & 5x_{n+\frac{1}{2}}^4 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 \\ 0 & 1 & 2x_{n+\frac{3}{2}} & 3x_{n+\frac{3}{2}}^2 & 4x_{n+\frac{3}{2}}^3 & 5x_{n+\frac{3}{2}}^4 \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 & 5x_{n+2}^4 \end{bmatrix}$$

the inverse of the matrix $C = (D^{-1})$ is computed using Maple 17 from which the following continuous scheme is obtained using (4) and Evaluating it at $x = x_n$, $x = x_{n+\frac{1}{2}}$, $x = x_{n+\frac{3}{2}}$ and $x = x_{n+2}$, the following discrete schemes are obtained

$$\left.\begin{aligned} y_n &= y_{n+1} - \frac{29}{180}hf_n - \frac{31}{45}hf_{n+\frac{1}{2}} - \frac{2}{15}hf_{n+1} - \frac{1}{45}hf_{n+\frac{3}{2}} + \frac{1}{180}hf_{n+2} \\ y_{n+\frac{1}{2}} &= y_{n+1} + \frac{19}{1440}hf_n - \frac{173}{720}hf_{n+\frac{1}{2}} - \frac{19}{60}hf_{n+1} + \frac{37}{120}hf_{n+\frac{3}{2}} - \frac{11}{180}hf_{n+2} \\ y_{n+\frac{3}{2}} &= y_{n+1} + \frac{11}{1440}hf_n - \frac{37}{720}hf_{n+\frac{1}{2}} + \frac{19}{60}hf_{n+1} + \frac{173}{720}hf_{n+\frac{3}{2}} - \frac{19}{1440}hf_{n+2} \\ y_{n+1} &= y_{n+1} - \frac{1}{180}hf_n + \frac{1}{45}hf_{n+\frac{1}{2}} + \frac{2}{15}hf_{n+1} + \frac{31}{45}hf_{n+\frac{3}{2}} + \frac{29}{180}hf_{n+2} \end{aligned}\right\} \tag{11}$$

For step number k = 3 incorporating one off-grid collocation points, here, the number of interpolation points, t = 1 and the number of collocation points, m = 5. Therefore, (4) becomes $y(x) = \alpha_2(x)y_{n+2} + h\beta_0(x)f_n + h\beta_1(x)f_{n+1} + h\beta_2(x)f_{n+2} + h\beta_{\frac{5}{2}}(x)f_{n+\frac{5}{2}} + h\beta_3(x)f_{n+3}$ Also the matrix *D*in (8) becomes

$$D=\begin{bmatrix} 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 \\ 0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 & 5x_{n+2}^4 \\ 0 & 1 & 2x_{n+\frac{5}{2}} & 3x_{n+\frac{5}{2}}^2 & 4x_{n+\frac{5}{2}}^3 & 5x_{n+\frac{5}{2}}^4 \\ 0 & 1 & 2x_{n+3} & 3x_{n+3}^2 & 4x_{n+3}^3 & 5x_{n+3}^4 \end{bmatrix}$$

The inverse of the matrix $C = (D^{-1})$ is computed using Maple 17 from which the following continuous scheme is also obtained using (4) and evaluating it at $x = x_n$, $x = x_{n+1}$, $x = x_{n+\frac{5}{2}}$ and $x = x_{n+3}$, The following discrete schemes are obtained

$$\left.\begin{aligned}
y_n &= y_{n+2} - \frac{71}{225}hf_n - \frac{64}{45}hf_{n+1} - \frac{1}{15}hf_{n+2} - \frac{64}{225}hf_{n+\frac{5}{2}} + \frac{4}{45}hf_{n+3} \\
y_{n+1} &= y_{n+2} + \frac{31}{1800}hf_n - \frac{151}{360}hf_{n+1} - \frac{109}{120}hf_{n+2} + \frac{88}{225}hf_{n+\frac{5}{2}} - \frac{29}{360}hf_{n+3} \\
y_{n+\frac{5}{2}} &= y_{n+2} + \frac{37}{28800}hf_n - \frac{67}{5760}hf_{n+1} + \frac{497}{1920}hf_{n+1} + \frac{61}{225}hf_{n+\frac{5}{2}} - \frac{113}{5760}hf_{n+3} \\
y_{n+3} &= y_{n+2} - \frac{1}{1800}hf_n + \frac{1}{360}hf_{n+1} + \frac{19}{120}hf_{n+2} + \frac{152}{225}hf_{n+\frac{5}{2}} + \frac{59}{360}hf_{n+3}
\end{aligned}\right\} \quad (12)$$

2.3 Convergence Analysis
Here the order, error constant, consistency and zero stability of the derived discrete schemes shall be investigated.

2.3.1 Order and Error Constant
The order and error constants of the discrete schemes in (10) are found in block form as follows

$$C_0 = \alpha_0 + \alpha_{\frac{1}{4}} + \alpha_{\frac{1}{2}} + \alpha_{\frac{3}{4}} + \alpha_1 = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$C_1 = \frac{1}{4}\alpha_{\frac{1}{4}} + \frac{1}{2}\alpha_{\frac{1}{2}} + \frac{3}{4}\alpha_{\frac{3}{4}} + \alpha_1 - (\beta_0 + \beta_{\frac{1}{4}} + \beta_{\frac{1}{2}} + \beta_{\frac{3}{4}} + \beta_1) = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$C_2 = \frac{1}{2!}\left(\left(\frac{1}{4}\right)^2\alpha_{\frac{1}{4}} + \left(\frac{1}{2}\right)^2\alpha_{\frac{1}{2}} + \left(\frac{3}{4}\right)^2\alpha_{\frac{3}{4}} + \alpha\right) - \frac{1}{1!}\left(\left(\frac{1}{4}\right)^1\beta_{\frac{1}{4}} + \left(\frac{1}{2}\right)^1\beta_{\frac{1}{2}} + \left(\frac{3}{4}\right)^1\beta_{\frac{3}{4}} + \beta\right) = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$C_3 = \frac{1}{3!}\left(\left(\frac{1}{4}\right)^3\alpha_{\frac{1}{4}} + \left(\frac{1}{2}\right)^3\alpha_{\frac{1}{2}} + \left(\frac{3}{4}\right)^3\alpha_{\frac{3}{4}} + \alpha\right) - \frac{1}{2!}\left(\left(\frac{1}{4}\right)^2\beta_{\frac{1}{4}} + \left(\frac{1}{2}\right)^2\beta_{\frac{1}{2}} + \left(\frac{3}{4}\right)^2\beta_{\frac{3}{4}} + \beta\right) = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$C_4 = \frac{1}{4!}\left(\left(\frac{1}{4}\right)^4\alpha_{\frac{1}{4}} + \left(\frac{1}{2}\right)^4\alpha_{\frac{1}{2}} + \left(\frac{3}{4}\right)^4\alpha_{\frac{3}{4}} + \alpha\right) - \frac{1}{3!}\left(\left(\frac{1}{4}\right)^3\beta_{\frac{1}{4}} + \left(\frac{1}{2}\right)^3\beta_{\frac{1}{2}} + \left(\frac{3}{4}\right)^3\beta_{\frac{3}{4}} + \beta\right) = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$C_5 = \frac{1}{5!}\left(\left(\frac{1}{4}\right)^5\alpha_{\frac{1}{4}} + \left(\frac{1}{2}\right)^5\alpha_{\frac{1}{2}} + \left(\frac{3}{4}\right)^5\alpha_{\frac{3}{4}} + \alpha\right) - \frac{1}{4!}\left(\left(\frac{1}{4}\right)^4\beta_{\frac{1}{4}} + \left(\frac{1}{2}\right)^4\beta_{\frac{1}{2}} + \left(\frac{3}{4}\right)^4\beta_{\frac{3}{4}} + \beta\right) = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$C_6=\frac{1}{6!}\left(\left(\frac{1}{4}\right)^6\alpha_{\frac{1}{4}}+\left(\frac{1}{2}\right)^6\alpha_{\frac{1}{2}}+\left(\frac{3}{4}\right)^6\alpha_{\frac{3}{4}}+\alpha\right)-\frac{1}{5!}\left(\left(\frac{1}{4}\right)^5\beta_{\frac{1}{4}}+\left(\frac{1}{2}\right)^5\beta_{\frac{1}{2}}+\left(\frac{3}{4}\right)^5\beta_{\frac{3}{4}}+\beta\right)=\begin{bmatrix}\frac{3}{655360}\\ \frac{1}{368640}\\ \frac{3}{655360}\\ 0\end{bmatrix}$$

Therefore, (10) has order, $p=5$ and error constants $=\frac{3}{655360},\frac{1}{368640},\frac{3}{655360},0$

Similarly, the order and error constants of the discrete schemes in (11) are found in block form as follows

$$C_0=\alpha_0+\alpha_{\frac{1}{2}}+\alpha_1+\alpha_{\frac{3}{2}}+\alpha_2=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$C_1=\frac{1}{2}\alpha_{\frac{1}{2}}+\alpha_1+\frac{3}{2}\alpha_{\frac{3}{2}}+2\alpha_2-(\beta_0+\beta_{\frac{1}{2}}+\beta_1+\beta_{\frac{3}{2}}+\beta_2)=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$C_2=\frac{1}{2!}\left(\left(\frac{1}{2}\right)^2\alpha_{\frac{1}{2}}+\alpha_1+\left(\frac{3}{2}\right)^2\alpha_{\frac{3}{2}}+(2)^2\alpha_2\right)-\frac{1}{1!}\left(\left(\frac{1}{2}\right)^1\beta_{\frac{1}{2}}+\beta_1+\left(\frac{3}{2}\right)^1\beta_{\frac{3}{2}}+(2)^1\beta_2\right)=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$C_3=\frac{1}{3!}\left(\left(\frac{1}{2}\right)^3\alpha_{\frac{1}{2}}+\alpha_1+\left(\frac{3}{2}\right)^3\alpha_{\frac{3}{2}}+(2)^3\alpha_2\right)-\frac{1}{2!}\left(\left(\frac{1}{2}\right)^2\beta_{\frac{1}{2}}+\beta_1+\left(\frac{3}{2}\right)^2\beta_{\frac{3}{2}}+(2)^2\beta_2\right)=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$C_4=\frac{1}{4!}\left(\left(\frac{1}{2}\right)^4\alpha_{\frac{1}{2}}+\alpha_1+\left(\frac{3}{2}\right)^4\alpha_{\frac{3}{2}}+(2)^4\alpha_2\right)-\frac{1}{3!}\left(\left(\frac{1}{2}\right)^3\beta_{\frac{1}{2}}+\beta_1+\left(\frac{3}{2}\right)^3\beta_{\frac{3}{2}}+(2)^3\beta_2\right)=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$C_5=\frac{1}{5!}\left(\left(\frac{1}{2}\right)^5\alpha_{\frac{1}{2}}+\alpha_1+\left(\frac{3}{2}\right)^5\alpha_{\frac{3}{2}}+(2)^5\alpha_2\right)-\frac{1}{4!}\left(\left(\frac{1}{2}\right)^4\beta_{\frac{1}{2}}+\beta_1+\left(\frac{3}{2}\right)^4\beta_{\frac{3}{2}}+(2)^4\beta_2\right)=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$C_6=\frac{1}{6!}\left(\left(\frac{1}{2}\right)^6\alpha_{\frac{1}{2}}+\alpha_1+\left(\frac{3}{2}\right)^6\alpha_{\frac{3}{2}}+(2)^6\alpha_2\right)-\frac{1}{5!}\left(\left(\frac{1}{2}\right)^5\beta_{\frac{1}{2}}+\beta_1+\left(\frac{3}{2}\right)^5\beta_{\frac{3}{2}}+(2)^5\beta_2\right)=\begin{bmatrix}-\frac{1}{5760}\\ \frac{11}{92160}\\ \frac{11}{92160}\\ -\frac{1}{5760}\end{bmatrix}$$

Therefore, (11) has order, $p=5$ and error constants $=-\frac{1}{5760},\frac{11}{92160},\frac{11}{92160},-\frac{1}{5760}$

Also, the order and error constants of the discrete schemes in (12) are found also in block form as follows

$$C_0=\alpha_0+\alpha_1+\alpha_2+\alpha_{\frac{5}{2}}+\alpha_3=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$C_1=\alpha_1+2\alpha_2+\frac{5}{2}\alpha_{\frac{5}{2}}+3\alpha_3-(\beta_0+\beta_1+\beta_2+\beta_{\frac{5}{2}}+\beta_3)=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$C_2=\frac{1}{2!}\left(\alpha_1+(2)^2\alpha_2+\left(\frac{5}{2}\right)^2\alpha_{\frac{5}{2}}+(3)^2\alpha_3\right)-\frac{1}{1!}\left(\beta_1+(2)^1\beta_2+\left(\frac{5}{2}\right)^1\beta_{\frac{5}{2}}+(3)^1\beta_3\right)=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$C_3=\frac{1}{3!}\left(\alpha_1+(2)^3\alpha_2+\left(\frac{5}{2}\right)^3\alpha_{\frac{5}{2}}+(3)^3\alpha_3\right)-\frac{1}{2!}\left(\beta_1+(2)^2\beta_2+\left(\frac{5}{2}\right)^2\beta_{\frac{5}{2}}+(3)^2\beta_3\right)=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$C_4=\frac{1}{4!}\left(\alpha_1+(2)^4\alpha_2+\left(\frac{5}{2}\right)^4\alpha_{\frac{5}{2}}+(3)^4\alpha_3\right)-\frac{1}{3!}\left(\beta_1+(2)^3\beta_2+\left(\frac{5}{2}\right)^3\beta_{\frac{5}{2}}+(3)^3\beta_3\right)=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$C_5=\frac{1}{5!}\left(\alpha_1+(2)^5\alpha_2+\left(\frac{5}{2}\right)^5\alpha_{\frac{5}{2}}+(3)^5\alpha_3\right)-\frac{1}{4!}\left(\beta_1+(2)^4\beta_2+\left(\frac{5}{2}\right)^4\beta_{\frac{5}{2}}+(3)^4\beta_3\right)=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$C_6 = \frac{1}{6!}\left(\alpha_1 + (2)^6 \alpha_2 + \left(\frac{5}{2}\right)^6 \alpha_{\frac{5}{2}} + (3)^6 \alpha_3\right) - \frac{1}{5!}\left(\beta_1 + (2)^5 \beta_2 + \left(\frac{5}{2}\right)^5 \beta_{\frac{5}{2}} + (3)^5 \beta_3\right) = \begin{bmatrix} -\frac{7}{900} \\ \frac{11}{3600} \\ \frac{83}{230400} \\ -\frac{1}{3600} \end{bmatrix}$$

Therefore, (12) has order, $p = 5$ and error constants $= -\frac{7}{900}, \frac{11}{360}, \frac{83}{230400}, -\frac{1}{360}$

2.3.2 Consistency
All the schemes in (10), (11) and (12) have their orders greater than one. So as in [12] the schemes are consistent.

2.3.3 Zero stability
The zero stability of the discrete schemes in (10) is determined in a block form as follows

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n+\frac{1}{4}} \\ y_{n+\frac{1}{2}} \\ y_{n+\frac{3}{4}} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n-\frac{3}{4}} \\ y_{n-\frac{1}{2}} \\ y_{n-\frac{1}{4}} \\ y_n \end{pmatrix}$$

$$+h \begin{pmatrix} \frac{232}{440} & \frac{-11}{120} & \frac{53}{1440} & \frac{-19}{2880} \\ \frac{31}{90} & \frac{1}{15} & \frac{1}{90} & \frac{-1}{360} \\ \frac{51}{160} & \frac{9}{40} & \frac{21}{160} & \frac{-3}{20} \\ \frac{16}{45} & \frac{2}{15} & \frac{16}{45} & \frac{7}{90} \end{pmatrix} \begin{pmatrix} f_{n+\frac{1}{4}} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{3}{4}} \\ f_{n+1} \end{pmatrix} + h \begin{pmatrix} 0 & 0 & 0 & \frac{251}{2800} \\ 0 & 0 & 0 & \frac{29}{360} \\ 0 & 0 & 0 & \frac{27}{320} \\ 0 & 0 & 0 & \frac{7}{9} \end{pmatrix} \begin{pmatrix} f_{n-\frac{3}{4}} \\ f_{n-\frac{1}{2}} \\ f_{n-\frac{1}{4}} \\ f_n \end{pmatrix}$$

Where $A_1^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, A_1^{(0)} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

And $B_1^{(1)} = \begin{pmatrix} \frac{232}{440} & \frac{-11}{120} & \frac{53}{1440} & \frac{-19}{2880} \\ \frac{31}{90} & \frac{1}{15} & \frac{1}{90} & \frac{-1}{360} \\ \frac{51}{160} & \frac{9}{40} & \frac{21}{160} & \frac{-3}{20} \\ \frac{16}{45} & \frac{2}{15} & \frac{16}{45} & \frac{7}{90} \end{pmatrix}$

The first characteristics polynomial of the block method of the discrete schemes in (2.9) is given by

$$p(\xi)=\det(\xi A_1^{(1)}-A_1^{(0)})=0$$
$$=\left|\xi A_1^{(1)}-A_1^{(0)}\right|$$
$= 0$

Now we have,

$$p(\xi)=\left|\xi\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}-\begin{pmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{pmatrix}\right|=\left|\begin{pmatrix}\xi&0&0&0\\0&\xi&0&0\\0&0&\xi&0\\0&0&0&\xi\end{pmatrix}-\begin{pmatrix}0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1\end{pmatrix}\right|$$

$$\Rightarrow p(\xi)=\begin{vmatrix}\xi&0&0&-1\\0&\xi&0&-1\\0&0&\xi&-1\\0&0&0&\xi-1\end{vmatrix}$$

Using Maple software we have,

$$p(\xi)=\xi^3(\xi-1)$$
$$\Rightarrow\ \xi^3(\xi-1)=0,$$
$\Rightarrow \xi_1=1,\ \xi_2=0,\ \xi_3=0,\ \xi_4=0,$ Since $\left|\xi_i\right|\le 1$, i = 1, 2, …, 4, then we observed that the discrete schemes in (10) satisfies the roots condition and hence it is zero stable.

Similarly, the zero stability of the discrete schemes in (11) is determined in block form as follows

$$\begin{pmatrix}0&-1&0&0\\1&-1&0&0\\0&-1&1&0\\0&-1&0&1\end{pmatrix}\begin{pmatrix}y_{n+\frac{1}{2}}\\y_{n+1}\\y_{n+\frac{3}{2}}\\y_{n+2}\end{pmatrix}=\begin{pmatrix}0&0&0&-1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}y_{n-\frac{3}{2}}\\y_{n-1}\\y_{n-\frac{1}{2}}\\y_n\end{pmatrix}$$

$$+h\begin{pmatrix}\frac{-31}{45}&\frac{-2}{15}&\frac{-1}{45}&\frac{9}{180}\\\frac{-173}{720}&\frac{-19}{60}&\frac{37}{120}&\frac{-11}{1440}\\\frac{-37}{720}&\frac{19}{60}&\frac{173}{720}&\frac{-19}{1440}\\\frac{1}{45}&\frac{2}{15}&\frac{31}{45}&\frac{29}{180}\end{pmatrix}\begin{pmatrix}f_{n+\frac{1}{2}}\\f_{n+1}\\f_{n+\frac{3}{2}}\\f_{n+2}\end{pmatrix}+h\begin{pmatrix}0&0&0&\frac{-29}{180}\\0&0&0&\frac{19}{1440}\\0&0&0&\frac{11}{1440}\\0&0&0&\frac{-1}{180}\end{pmatrix}\begin{pmatrix}f_{n-\frac{3}{2}}\\f_{n-1}\\f_{n-\frac{1}{2}}\\f_n\end{pmatrix}$$

Where $A_2^{(1)}=\begin{pmatrix}0&-1&0&0\\1&-1&0&0\\0&-1&1&0\\0&-1&0&1\end{pmatrix},A_2^{(0)}=\begin{pmatrix}0&0&0&-1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$

And $B_2^{(1)} = \begin{pmatrix} \frac{-31}{45} & \frac{-2}{15} & \frac{-1}{45} & \frac{1}{180} \\ \frac{-173}{720} & \frac{-19}{60} & \frac{37}{120} & \frac{-11}{1440} \\ \frac{-37}{720} & \frac{19}{60} & \frac{173}{720} & \frac{-19}{1440} \\ \frac{1}{45} & \frac{2}{15} & \frac{31}{45} & \frac{29}{180} \end{pmatrix}$

The first characteristics polynomial of the block method of the discrete schemes in (11) is given by

$$p(\xi) = \det(\xi A_2^{(1)} - A_2^{(0)}) = 0$$

$$= \left|\xi A_2^{(1)} - A_2^{(0)}\right|$$

$= 0$

Now we have,

$$p(\xi) = \left| \xi \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right| = \left| \begin{pmatrix} 0 & -\xi & 0 & 0 \\ \xi & -\xi & 0 & 0 \\ 0 & -\xi & \xi & 0 \\ 0 & -\xi & 0 & \xi \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right|$$

$$\Rightarrow p(\xi) = \begin{vmatrix} 0 & -\xi & 0 & 1 \\ 0 & -\xi & 0 & 0 \\ 0 & -\xi & \xi & 0 \\ 0 & -\xi & 0 & \xi \end{vmatrix}$$

Using Maple software we have,

$$p(\xi) = -\xi^3(-\xi + 1)$$

$$\Rightarrow -\xi^3(-\xi + 1) = 0,$$

$\Rightarrow \xi_1 = 1, \ \xi_2 = 0, \ \xi_3 = 0, \ \xi_4 = 0,$ Since $\left|\xi_i\right| \leq 1$, i = 1, 2, 4, we observed that the discrete schemes in (11) satisfies the roots condition and hence it is zero stable.

Also, the zero stability of the discrete schemes in (12) is determined in block form as follows

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+\frac{5}{2}} \\ y_{n+3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{n-\frac{5}{2}} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix}$$

$$+h\begin{pmatrix} \frac{-64}{45} & \frac{-1}{15} & \frac{-64}{225} & \frac{4}{45} \\ \frac{-151}{360} & \frac{-109}{102} & \frac{88}{225} & \frac{-29}{360} \\ \frac{-67}{5760} & \frac{497}{1920} & \frac{61}{225} & \frac{-113}{5760} \\ \frac{1}{360} & \frac{19}{120} & \frac{152}{225} & \frac{59}{360} \end{pmatrix}\begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+\frac{5}{2}} \\ f_{n+3} \end{pmatrix} + h\begin{pmatrix} 0 & 0 & 0 & \frac{-71}{225} \\ 0 & 0 & 0 & \frac{31}{1800} \\ 0 & 0 & 0 & \frac{37}{28800} \\ 0 & 0 & 0 & \frac{-1}{1800} \end{pmatrix}\begin{pmatrix} f_{n-\frac{5}{2}} \\ f_{n-2} \\ f_{n-1} \\ f_{n} \end{pmatrix}$$

Where $A_3^{(1)} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}, A_3^{(0)} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

And $B_3^{(1)} = \begin{pmatrix} \frac{-64}{45} & \frac{-1}{15} & \frac{-64}{225} & \frac{4}{45} \\ \frac{-151}{360} & \frac{-109}{102} & \frac{88}{225} & \frac{-29}{360} \\ \frac{-67}{5760} & \frac{497}{1920} & \frac{61}{225} & \frac{-113}{5760} \\ \frac{1}{360} & \frac{19}{120} & \frac{152}{225} & \frac{59}{360} \end{pmatrix}$

The first characteristics polynomial of the block method of the discrete schemes in (12) is given by

$$p(\xi) = \det(\xi A_3^{(1)} - A_3^{(0)}) = 0$$

$$= \left|\xi A_3^{(1)} - A_3^{(0)}\right|$$

= 0

Now we have,

$$p(\xi) = \left| \xi\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right| = \left| \begin{pmatrix} 0 & -\xi & 0 & 0 \\ \xi & -\xi & 0 & 0 \\ 0 & -\xi & \xi & 0 \\ 0 & -\xi & 0 & \xi \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right|$$

$$\Rightarrow p(\xi) = \begin{vmatrix} 0 & -\xi & 0 & 1 \\ 0 & -\xi & 0 & 0 \\ 0 & -\xi & \xi & 0 \\ 0 & -\xi & 0 & \xi \end{vmatrix}$$

Using Maple software we have,

$$p(\xi) = -\xi^3(-\xi+1)$$

$$\Rightarrow -\xi^3(-\xi+1) = 0,$$

$\Rightarrow \xi_1 = 1,\ \xi_2 = 0,\ \xi_3 = 0,\ \xi_4 = 0,$ Since $|\xi_i| \le 1$, i = 1, 2, 4, then by [12] we observed that the discrete schemes in (2.17) satisfies the roots condition and hence it is zero stable.

2.3.4 Convergence

The block discrete schemes methods in (10), (11) and (12) are convergent since they are consistent and zero-stable.

2.4 Region of Absolute Stability

The region of absolute of the discrete schemes in (10) can be found in a block form as follows

Let $f_z = \det(r(A_1^{(1)} - zB_1^{(1)}) - A_1^{(0)})$ and $f_{zp} =$ derivative of f_z

Where $A_1^{(1)}$, $A_1^{(0)}$ and $B_1^{(1)}$ were obtained in the above

Using MATLAB, the region of absolute stability of the discrete schemes in (10) is plotted as shown in Fig. 1

Similarly, the region of absolute of the discrete schemes in (11) can be found in a block form as follows

Let $f_z = \det(r(A_2^{(1)} - zB_2^{(1)}) - A_2^{(0)})$ and $f_{zp} =$ derivative of f_z

Where $A_2^{(1)}$, $A_2^{(0)}$ and $B_2^{(1)}$ were obtained in the above

Using MATLAB, the region of absolute stability of the discrete schemes in (11) is plotted as shown in Fig. 2

Also, the region of absolute of the discrete schemes in (12) can be found in a block form as follows:

Let $f_z = \det(r(A_3^{(1)} - zB_3^{(1)}) - A_3^{(0)})$ and $f_{zp} =$ derivative of f_z

Where $A_3^{(1)}$, $A_3^{(0)}$ and $B_3^{(1)}$ were obtained in the above

Using MATLAB, the region of absolute stability of the discrete schemes in (12) is plotted as shown in Fig. 3

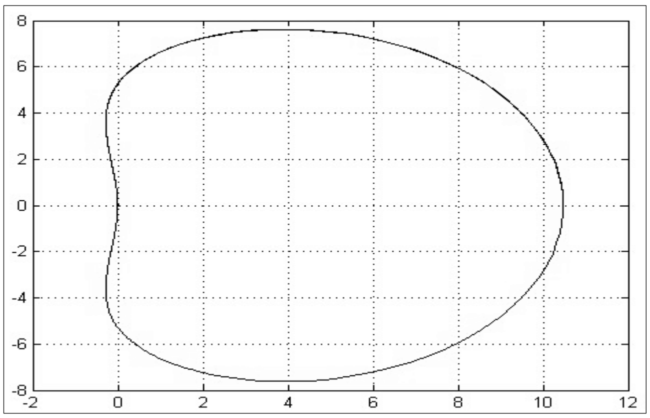

Fig 1: Region of absolute stability HAMM (10)

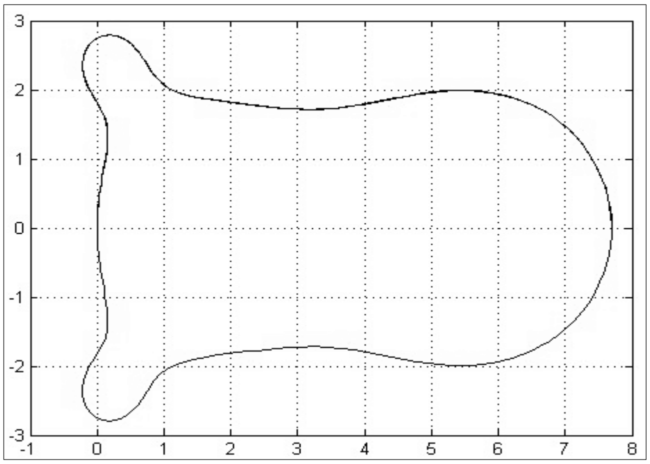

Fig 2: Region of absolute stability HAMM (11)

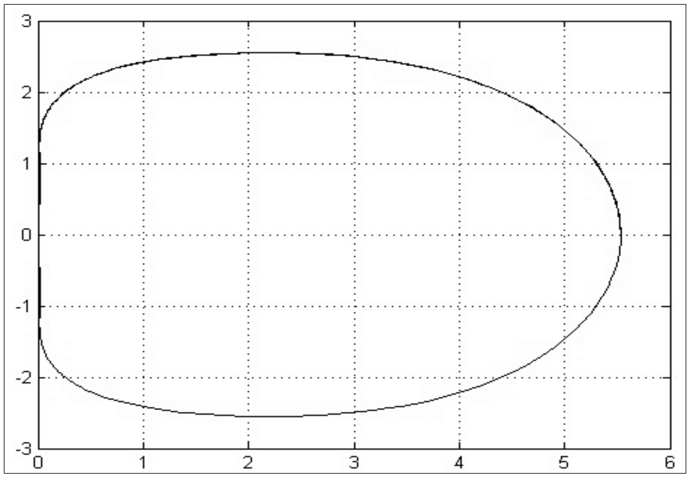

Fig 3: Region of absolute stability HAMM (12)

Note that in Fig. 1 and 2, we observed that both HAMM (10) and (11) are $A(\alpha)$ stable, while in Fig. 3, we observed that HAMM (12) is A-stable.

3. Numerical Computations

In this section, some stiff initial value problems shall be solved using the block form of the discrete schemes derived in previous section.

3.1 Numerical Examples

Example 1

$$y' = \begin{bmatrix} -21 & 19 & -20 \\ 19 & -21 & 20 \\ 40 & -40 & -40 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$y(0) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad h = 0.3, \quad 0 \le x \le 3,$$

$$\text{Exact solution} \quad y(x) = \frac{1}{2} \begin{bmatrix} e^{-2x} + e^{-40x}\left(cos(40x) + sin(40x)\right) \\ e^{-2x} - e^{-40x}\left(cos(40x) + sin(40x)\right) \\ 2e^{-40x}\left(\sin(40x) - \cos(40x)\right) \end{bmatrix}$$

Solution

Using (10) the schemes in above example are obtained for $j=1$, $f_n = -21y_{1,n} + 19y_{2,n} - 20y_{3,n}$, for $j=2$, $f_n = 19y_{1,n} - 21y_{2,n} + 20y_{3,n}$ and for $j=3$, $f_n = 40y_{1,n} - 40y_{2,n} - 40y_{3,n}$ substituting $h=0.3$ and corresponding values of f_n's, the following are obtained. The results of the above are obtained in block form using Maple 17 varying *n* = 0, 1, 2… 29 and evaluating the values of y_n. The results are summarized in the table below.

Table 1: Solution of Example 1 using the HAMM for step number *k* = 1.

x	Exact Solution y_1	Numerical Solution y_1	Exact Solution y_2	Numerical Solution y_2	Exact Solution y_3	Numerical Solution y_3
0.3	2.74E-01	2.74E-01	2.74E-01	2.74E-01	-8.48E-06	3.15E-05
0.6	1.51E-01	1.51E-01	1.51E-01	1.51E-01	-5.02E-11	2.60E-10
0.9	8.26E-02	8.26E-02	8.26E-02	8.26E-02	-2.00E-16	1.00E-11
1.2	4.54E-02	4.54E-02	4.54E-02	4.54E-02	-1.83E-22	1.00E-11

1.5	2.49E-02	2.49E-02	2.49E-02	2.49E-02	5.67E-27	7.00E-12
1.8	1.37E-02	1.37E-02	1.37E-02	1.37E-02	6.57E-32	4.10E-11
2.1	7.50E-03	7.50E-03	7.50E-03	7.50E-03	4.67E-37	4.90E-12
2.4	4.11E-03	4.11E-03	4.11E-03	4.11E-03	2.36E-42	3.00E-13
2.7	2.26E-03	2.26E-03	2.26E-03	2.26E-03	6.88E-48	1.80E-12
3.0	1.24E-03	1.24E-03	1.24E-03	1.24E-03	-1.79E-53	4.00E-13

Similarly, using (11) the schemes in above example are obtained

For $j=1$, $f_n=-21y_{1,n}+19y_{2,n}-20y_{3,n}$, for $j=2$, $f_n=19y_{1,n}-21y_{2,n}+20y_{3,n}$ and for $j=3$, $f_n=40y_{1,n}-40y_{2,n}-40y_{3,n}$ substituting $h=0.3$ and corresponding values of $f_n's$, the following are obtained

The results of the above are obtained in block form using Maple 17 varying

n = 0, 2, 4, 28 and evaluating the values of y_n. The results are summarized in the table below.

Table 2: Solution of Example 1 using the HAMM for step number k = 2

x	Exact Solution y_1	Numerical Solution y_1	Exact Solution y_2	Numerical Solution y_2	Exact Solution y_3	Numerical Solution y_3
0.3	2.74E-01	2.74E-01	2.74E-01	2.74E-01	1.89E-06	6.68E-02
0.6	1.51E-01	1.52E-01	1.51E-01	1.50E-01	-1.82E-11	-4.36E-04
0.9	8.26E-02	8.26E-02	8.26E-02	8.27E-02	-2.60E-16	1.07E-05
1.2	4.54E-02	4.54E-02	4.54E-02	4.54E-02	-2.01E-21	7.62E-07
1.5	2.49E-02	2.49E-02	2.49E-02	2.49E-02	-1.10E-26	7.72E-09
1.8	1.37E-02	1.37E-02	1.37E-02	1.37E-02	-3.84E-32	2.49E-09
2.1	7.50E-03	7.50E-03	7.50E-03	7.50E-03	1.76E-38	-1.91E-12
2.4	4.11E-03	4.11E-03	4.11E-03	4.11E-03	1.63E-42	1.40E-11
2.7	2.26E-03	2.26E-03	2.26E-03	2.26E-03	1.63E-47	9.99E-13
3.0	1.24E-03	1.24E-03	1.24E-03	1.24E-03	1.07E-52	1.70E-12

Also, using (12) the schemes in the above example are obtained

For $j=1$, $f_n=-21y_{1,n}+19y_{2,n}-20y_{3,n}$, for $j=2$, $f_n=19y_{1,n}-21y_{2,n}+20y_{3,n}$ and for $j=3$, $f_n=40y_{1,n}-40y_{2,n}-40y_{3,n}$ substituting $h=0.3$ and corresponding values of $f_n's$, the following are obtained. The results of the above are obtained in block form using Maple 17 varying

n = 0, 3, 6, 27 and evaluating the values of y_n. The results are summarized in the table below.

Table 3: Solution of Example 1 using the HAMM for step number k = 3

x	Exact Solution y_1	Numerical Solution y_1	Exact Solution y_2	Numerical Solution y_2	Exact Solution y_3	Numerical Solution y_3
0.3	2.74E-01	2.56E-01	2.74E-01	2.93E-01	1.89E-06	-4.24E-02
0.6	1.51E-01	1.50E-01	1.51E-01	1.51E-01	-1.82E-11	1.38E-03
0.9	8.26E-02	8.27E-02	8.26E-02	8.26E-02	-2.60E-16	7.47E-05
1.2	4.54E-02	4.54E-02	4.54E-02	4.54E-02	-2.01E-21	-1.82E-06
1.5	2.49E-02	2.49E-02	2.49E-02	2.49E-02	-1.10E-26	-1.28E-07
1.8	1.37E-02	1.37E-02	1.37E-02	1.37E-02	-3.84E-32	2.25E-09
2.1	7.50E-03	7.50E-03	7.50E-03	7.50E-03	1.76E-38	2.26E-10
2.4	4.11E-03	4.11E-03	4.11E-03	4.11E-03	1.63E-42	-5.00E-12
2.7	2.26E-03	2.26E-03	2.26E-03	2.26E-03	1.63E-47	3.10E-12
3.0	1.24E-03	1.24E-03	1.24E-03	1.24E-03	1.07E-52	-2.00E-13

Example 2

$$y' = \begin{bmatrix} -0.1 & -49.9 & 0 \\ 0 & -50 & 0 \\ 0 & 70 & -120 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$y(0) = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \quad h = 0.3, \quad 0 \le x \le 3,$$

$$\text{Exact solution} \quad y(x) = \begin{bmatrix} e^{-0.1x} + e^{-50x} \\ e^{-50x} \\ e^{-50x} + e^{-120x} \end{bmatrix}$$

Solution

Using (10) the schemes in the above example are obtained for $j=1$, $f_n = -0.1y_{1,n} - 49.9y_{2,n}$, for $j=2$, $f_n = -50y_{2,n}$ and for $j=3$, $f_n = 70y_{2,n} - 120y_{3,n}$ substituting $h=0.3$ and corresponding values of $f_n's$, the following are obtained The results of the above are obtained in block form using Maple 17 varying n = 0, 1, 2, …, 29 and evaluating the values of y_n.The results are summarized in the table below.

Table 4: Solution of Example 2 using the HAMM for step number k = 1

x	Exact Solution y_1	Numerical Solution y_1	Exact Solution y_2	Numerical Solution y_2	Exact Solution y_3	Numerical Solution y_3
0.3	9.70E-01	9.70E-01	3.06E-07	1.47E-06	3.06E-07	3.81E-04
0.6	9.42E-01	9.42E-01	9.36E-14	2.18E-12	9.36E-14	1.44E-07
0.9	9.14E-01	9.14E-01	2.86E-20	3.21E-18	2.86E-20	5.48E-11
1.2	8.87E-01	8.87E-01	8.76E-27	4.73E-24	8.76E-27	2.08E-14
1.5	8.61E-01	8.61E-01	2.68E-33	6.98E-30	2.68E-33	7.90E-18
1.8	8.35E-01	8.35E-01	8.19E-40	1.03E-35	8.19E-40	3.00E-21
2.1	8.11E-01	8.11E-01	2.51E-46	1.52E-41	2.51E-46	1.14E-24
2.4	7.87E-01	7.87E-01	7.67E-53	2.24E-47	7.67E-53	4.33E-28
2.7	7.63E-01	7.63E-01	2.35E-59	3.30E-53	2.35E-59	1.64E-31
3.0	7.41E-01	7.41E-01	7.18E-66	4.87E-59	7.18E-66	6.24E-35

Similarly using (11) the schemes in the above example are obtained for $j=1$, $f_n = -0.1y_{1,n} - 49.9y_{2,n}$, for $j=2$, $f_n = -50y_{2,n}$ and for $j=3$, $f_n = 70y_{2,n} - 120y_{3,n}$ substituting $h=0.3$ and corresponding values of $f_n's$, the following are obtained The results of the above are obtained in block form using Maple 17 varying n = 0, 2, 4, 28 and evaluating the values of y_n. The results are summarized in the table below.

Table 5: Solution of Example 2 using the HAMM for step number k = 2

x	Exact Solution y_1	Numerical Solution y_1	Exact Solution y_2	Numerical Solution y_2	Exact Solution y_3	Numerical Solution y_3
0.3	9.70E-01	9.71E-01	3.06E-07	8.96E-04	3.06E-07	1.83E-02
0.6	9.42E-01	9.42E-01	9.36E-14	9.59E-05	9.36E-14	1.66E-02
0.9	9.14E-01	9.14E-01	2.86E-20	8.60E-08	2.86E-20	2.87E-04
1.2	8.87E-01	8.87E-01	8.76E-27	9.21E-09	8.76E-27	2.73E-04
1.5	8.61E-01	8.61E-01	2.68E-33	8.25E-12	2.68E-33	4.75E-06
1.8	8.35E-01	8.35E-01	8.19E-40	8.83E-13	8.19E-40	4.50E-06
2.1	8.11E-01	8.11E-01	2.51E-46	7.92E-16	2.51E-46	7.84E-08
2.4	7.87E-01	7.87E-01	7.67E-53	8.48E-17	7.67E-53	7.44E-08
2.7	7.63E-01	7.63E-01	2.35E-59	7.60E-20	2.35E-59	1.29E-09
3.0	7.41E-01	7.41E-01	7.18E-66	8.13E-21	7.18E-66	1.23E-09

Also using (12) the schemes in the above example are obtained for $j=1$, $f_n=-0.1y_{1,n}-49.9y_{2,n}$, for $j=2$, $f_n=-50y_{2,n}$ and for $j=3$, $f_n=70y_{2,n}-120y_{3,n}$ substituting $h=0.3$ and corresponding values of $f_n's$, the following are obtained The results of the above are obtained in block form using Maple 17 varying

n = 0, 3, 6, 27 and evaluating the values of y_n. The results are summarized in the table below.

Table 6: Solution of Example 2 using the HAMM for step number k = 3

x	Exact Solution y_1	Numerical Solution y_1	Exact Solution y_2	Numerical Solution y_2	Exact Solution y_3	Numerical Solution y_3
0.3	9.70E-01	9.90E-01	3.06E-07	1.93E-02	3.06E-07	9.19E-02
0.6	9.42E-01	9.42E-01	9.36E-14	3.73E-04	9.36E-14	5.64E-03
0.9	9.14E-01	9.14E-01	2.86E-20	-6.03E-07	2.86E-20	3.90E-04
1.2	8.87E-01	8.87E-01	8.76E-27	-1.16E-08	8.76E-27	2.79E-05
1.5	8.61E-01	8.61E-01	2.68E-33	-2.25E-10	2.68E-33	2.02E-06
1.8	8.35E-01	8.35E-01	8.19E-40	-4.34E-12	8.19E-40	1.47E-07
2.1	8.11E-01	8.11E-01	2.51E-46	-8.38E-14	2.51E-46	1.06E-08
2.4	7.87E-01	7.87E-01	7.67E-53	-1.62E-15	7.67E-53	7.73E-10
2.7	7.63E-01	7.63E-01	2.35E-59	-3.12E-17	2.35E-59	5.61E-11
3.0	7.41E-01	7.41E-01	7.18E-66	7.20E-18	7.18E-66	4.07E-12

4. Results and Discussions

In this section, the performances of the schemes derived in chapter two shall be measured in solving the examples in chapter three by computing their absolute errors.

4.1 Analysis of Results

The analysis of results is obtained by evaluating absolute difference of exact solutions and numerical solutions. The results are summarized in the following tables

Table 7: Absolute errors of HAMM for step number k = 1, using Example 1

x	Error y_1	Error y_2	Error y_3
0.3	1.07E-05	1.07E-05	4.00E-05
0.6	1.53E-10	1.03E-09	3.10E-10
0.9	3.09E-10	4.69E-10	1.00E-11
1.2	2.75E-10	2.85E-10	1.00E-11
1.5	2.36E-10	2.16E-10	7.00E-12
1.8	1.36E-10	1.56E-10	4.10E-11
2.1	6.68E-11	5.88E-11	4.90E-12
2.4	3.15E-11	3.25E-11	3.00E-13
2.7	2.37E-11	2.17E-11	1.80E-12
3.0	1.27E-11	1.27E-11	4.00E-13

Table 8: Absolute errors of HAMM for step number k = 2, using Example 1

x	Error y_1	Error y_2	Error y_3
0.3	7.84E-05	7.84E-05	6.68E-02
0.6	9.08E-04	9.08E-04	4.36E-04
0.9	3.17E-06	3.17E-06	1.07E-05
1.2	1.17E-06	1.17E-06	7.62E-07
1.5	6.62E-09	7.94E-09	7.72E-09
1.8	1.67E-09	4.44E-10	2.49E-09
2.1	2.96E-10	3.21E-10	1.91E-12
2.4	2.45E-10	2.42E-10	1.40E-11
2.7	1.26E-10	1.26E-10	9.99E-13
3.0	9.17E-11	9.17E-11	1.70E-12

Table 9: Absolute errors of HAMM for step number k = 3, using Example 1

x	Error y_1	Error y_2	Error y_3
0.3	1.87E-02	1.87E-02	4.24E-02
0.6	8.93E-04	8.93E-04	1.38E-03
0.9	2.51E-05	2.53E-05	7.47E-05

1.2	1.49E-06	1.61E-06	1.82E-06
1.5	7.32E-08	8.76E-09	1.28E-07
1.8	2.96E-08	2.44E-08	2.25E-09
2.1	1.72E-08	1.73E-08	2.26E-10
2.4	1.08E-08	1.09E-08	5.00E-12
2.7	6.70E-09	6.70E-09	3.10E-12
3.0	4.08E-09	4.08E-09	2.00E-13

Table 10: Absolute errors of HAMM for step number k = 1, using Example 2

x	Error y_1	Error y_2	Error y_3
0.3	1.17E-06	1.17E-06	3.81E-04
0.6	1.18E-09	2.08E-12	1.44E-07
0.9	1.57E-09	3.18E-18	5.48E-11
1.2	2.22E-09	4.72E-24	2.08E-14
1.5	2.73E-09	6.98E-30	7.90E-18
1.8	3.11E-09	1.03E-35	3.00E-21
2.1	3.47E-09	1.52E-41	1.14E-24
2.4	3.87E-09	2.24E-47	4.33E-28
2.7	4.04E-09	3.30E-53	1.64E-31
3.0	4.48E-09	4.87E-59	6.24E-35

Table 11: Absolute errors of the HAMM for step number k = 2, using Example 2

x	Error y_1	Error y_2	Error y_3
0.3	8.96E-04	8.96E-04	1.83E-02
0.6	9.59E-05	9.59E-05	1.66E-02
0.9	8.62E-08	8.60E-08	2.87E-04
1.2	9.38E-09	9.21E-09	2.73E-04
1.5	2.75E-10	8.25E-12	4.75E-06
1.8	4.89E-10	8.83E-13	4.50E-06
2.1	6.30E-10	7.92E-16	7.84E-08
2.4	8.33E-10	8.48E-17	7.44E-08
2.7	8.63E-10	7.60E-20	1.29E-09
3.0	9.18E-10	8.13E-21	1.23E-09

Table 12: Absolute errors of HAMM for step number k = 3, using Example 2

x	Error y_1	Error y_2	Error y_3
0.3	1.93E-02	1.93E-02	9.19E-02
0.6	3.73E-04	3.73E-04	5.64E-03
0.9	7.19E-06	6.03E-07	3.90E-04
1.2	1.34E-07	1.16E-08	2.79E-05
1.5	2.63E-09	2.25E-10	2.02E-06
1.8	6.21E-09	4.34E-12	1.47E-07
2.1	7.07E-09	8.38E-14	1.06E-08
2.4	7.77E-09	1.62E-15	7.73E-10
2.7	8.34E-09	3.12E-17	5.61E-11
3.0	8.88E-09	7.20E-18	4.07E-12

4.2 Conclusion

In conclusion, the discrete schemes of the HAMM with three, two and one off-grid collocation points for step number k = 1, 2 and 3 respectively were deduced from their continuous schemes.

It was observed that all are convergent and for step number k = 1 and 2 are $A(\alpha)$-stable while for step number k = 3 is A-stable. It was also observed from the error tables that the HAMM schemes for step number k = 3 incorporating one off-grid collocation point performed better than the HAMM schemes for step number k = 2 and 1 incorporating two and three off-grid collocation points respectively in solving a highly stiff initial value problem as seen in Example 1 and 2 in the above section

4.3 Recommendations

It is recommended the HAMM schemes with more off-grid points for a lower step number performed less well than the HAMM schemes with less off-grid points for a higher step number in solving highly stiff initial value problems.

4.4 Recommendations for Further Research

It is recommended that further research should be carried out on the derivation of continuous and discrete schemes for HAMM incorporating one or more off-grid collocation points for step numbers $k = 4, 5, \ldots$ and their application for the solution of linear and nonlinear problems.

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