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The Solutions of Selected Problems of the IMO

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Abstract
Using the elementary theory of the inequality, the paper obtains all the real numbers x which satisfy the inequality of the problem 1; on the basis of the expanding formula of cosine of the angle 2x, it explicitly solves the equation of problem 2, and obtains all the solutions which satisfy the equation of the problem; it proves the equality of the problem 3 by use of the mathematical induction; moreover, it solves and obtains all the solutions (x_1,x_2,x_3,x_4) of problem 4; using the mathematical induction, the paper proves the conclusion of the problem 5; and finally, the paper offers a simple and rigorous proof on the problem 6.

Keywords: The IMO, problems 1-6, inequality, equality, $Cos2x$, the mathematical induction, fraction

1. Introduction
The international mathematical Olympiad(IMO) has, beginning from 1959, held 60 sessions. In the first a few sessions, perhaps because the mathematics teaching all over the world was backward, and the problems of IMO are rather difficult [1], the highest score obtained by the champion is very low. For example, the 4th IMO held in Czechoslovakia in 1962, although the Hungary won the championship, its total score is merely 7 points. Therefore, the IMO attracts more and more top pupils in high schools and other peoples who are interesting in mathematics to participate and promote the activities of IMO [2-14]. In general, there are six problems in every session of the IMO, with the intention of improving the abilities of solving the problems of the IMO, this paper will also discuss and solve six problems selected from the first to 11th IMO.

2. Problem 1
Problem 1(4th IMO) [15]: Determining all the real numbers x which satisfy the inequality: $\sqrt{3-x}-\sqrt{x+1}>\frac{1}{2}$.

Solution: At first, $\sqrt{3-x}$ and $\sqrt{x+1}$ restrict that x must satisfy

$$-1\le x\le 3 \qquad (1).$$

On the other hand, from

$$\sqrt{3-x}-\sqrt{x+1}>\frac{1}{2} \qquad (2),$$

it arrives

$$(\sqrt{3-x}-\sqrt{x+1})^2>\frac{1}{4} \qquad (3),$$

and

$$2-\sqrt{(3-x)(x+1)}>\frac{1}{8} \qquad (4),$$

therefore,

$$\sqrt{(3-x)(x+1)}<\frac{15}{8} \qquad (5),$$

and

$$(3-x)(x+1)<\frac{225}{64} \qquad (6).$$

Eq.(6) can be changed into

$$(x-1)^2 > \frac{31}{64} \tag{7}$$

hence, for the positive square roots of the two sides of the inequality (7), to ensure $\sqrt{(x-1)^2} > 0$, it holds true that

$$\pm(x-1) > \frac{\sqrt{31}}{8} \tag{8}$$

In consideration of eq.(8), in case of $+(x-1) > \frac{\sqrt{31}}{8}$,

$$x > 1 + \frac{\sqrt{31}}{8} \tag{9}$$

but if so, $\sqrt{3-x} - \sqrt{x+1} < 0$, it contradicts with eq.(2).

In case of $-(x-1) > \frac{\sqrt{31}}{8}$, it results in

$$x < 1 - \frac{\sqrt{31}}{8} \tag{10}$$

Eq.(10) is not contrary to eq.(2), therefore, considering eq.(1) and eq.(10), the solution of question 2 of the 4th IMO is given by

$$-1 \le x < 1 - \frac{\sqrt{31}}{8} \tag{11}$$

3. Problem 2

Problem 2(4th IMO) [15]: Solve equation: $Cos^2 x + Cos^2 2x + Cos^2 3x = 1$.

Solution:

$$Cos^2 2x = (Cos^2 x - Sin^2 x)^2 = (2Cos^2 x - 1)^2 = 4Cos^4 x - 4Cos^2 x + 1 \tag{12}$$

$$Cos^2 3x = (Cos2xCosx - 2Sin^2 xCosx)^2 = Cos^2 x(4Cos^2 x - 3)^2 \tag{13}$$

Therefore, the equation in problem 2 can be changed into

$$Cos^2 x[4Cos^2 x - 3 + (4Cos^2 x - 3)^2] = 0 \tag{14}$$

hence

$$Cos^2 x(4Cos^2 x - 3)(4Cos^2 x - 2)] = 0 \tag{15}$$

From eq.(15) it arrives:

$$Cos^2 x = 0, \quad x = \pm\frac{\pi}{2}; \tag{16}$$

$$4Cos^2 x - 2 = 0, \quad x = \pm\frac{\pi}{4}; \tag{17}$$

$$4Cos^2 x - 3 = 0, \quad x = \pm\frac{\pi}{6}. \tag{18}$$

Therefore, the solution of problem 2 is:

$$x = 2k\pi \pm \frac{\pi}{2}\,;\ x = 2k\pi \pm \frac{\pi}{4}\,;\ x = 2k\pi \pm \frac{\pi}{6}\quad (k=0,\pm 1,\pm 2,\pm 3,\cdots\ldots) \qquad (19).$$

4. Problem 3

Problem 3(8th IMO)[15]: Prove the following equality:

$$\sum_{t=1}^{n}\frac{1}{Sin2^{t}x} = Cotx - Cot2^{n}x$$
,

where $n\in N$ and $x\notin \pi Z/2^{t}$ for every $t\in N$.

Solution: In order to prove the problem

$$\sum_{t=1}^{n}\frac{1}{Sin2^{t}x} = Cotx - Cot2^{n}x \qquad (20),$$

Using the mathematical induction, when n=1,

$$\sum_{t=1}^{n}\frac{1}{Sin2^{t}x} = \frac{1}{Sin2x} \qquad (21),$$

and

$$Cotx - Cot2^{n}x = \frac{Cosx}{Sinx} - \frac{Cos2x}{Sin2x} = \frac{2Cos^{2}x - Cos2x}{Sin2x} = \frac{1}{Sin2x} \qquad (22)$$

Comparing eq.(22) with eq.(21), eq.(20) is true.

Supposing when n=k, $\sum_{t=1}^{k}\frac{1}{Sin2^{t}x} = Cotx - Cot2^{k}x$, thus when n=k+1,

$$\sum_{t=1}^{k+1}\frac{1}{Sin2^{t}x} = \sum_{t=1}^{k}\frac{1}{Sin2^{k}x} + \frac{1}{Sin2^{k+1}x} = Cotx - Cot2^{k}x + \frac{1}{Sin2^{k+1}x}$$
$$= Cotx - \frac{Cos2^{k}x}{Sin2^{k}x} + \frac{1}{2Sin2^{k}xCos2^{k}x} = Cotx + \frac{1-2Cos^{2}2^{k}x}{2Sin2^{k}xCos2^{k}x} \qquad (23),$$
$$= Cotx - \frac{Cos2^{k+1}x}{Sin2^{k+1}x} = Cotx - Cot2^{k+1}x$$

therefore, eq.(20) is true

.

5. Problem 4

Problem 4(7th IMO) [15]: Find four real numbers x_1, x_2, x_3, x_4 such that the sum of any of the numbers and the product of other three is equal to 2.

Solution: In accord with the meaning of the problem, there are following a group of equations:

$$x_1 + x_2x_3x_4 = 2 \qquad (24),$$

$$x_2 + x_1x_3x_4 = 2 \qquad (25),$$

$$x_3 + x_1x_2x_4 = 2 \qquad (26),$$

$$x_4 + x_1x_2x_3 = 2 \qquad (27).$$

From (24) - (25) and (26)-(27) it obtains:

$$(x_1 - x_2)(1 - x_3x_4) = 0 \qquad (28),$$

$$(x_3 - x_4)(1 - x_1 x_2) = 0 \quad (29).$$

If $x_1 - x_2 = 0$, $x_3 - x_4 = 0$, and $1 - x_3 x_4 = 0$, $1 - x_1 x_2 = 0$, it can be obtained that $x_1 = x_2 = \pm 1$; $x_3 = x_4 = \pm 1$. If $x_1 = x_3 = +1$, the first solution of the equations is $(x_1, x_2, x_3, x_4) = (1,1,1,1)$.
If $x_3 = x_4 = -1$, eq.(24) becomes $x_1 + x_2 = 2$, substitute it into eq.(26), it obtains

$$x_1^2 - 2x_1 - 3 = 0 \quad (30),$$

the solutions of eq.(30) are

$$x_1 = \frac{2 \pm \sqrt{(-2)^2 - 4 \times 1 \times (-3)}}{2} \quad (31),$$

if $x_1 = 3$, it is easy to get $x_2 = -1$; if x_1 take the value of -1, it is easy to get $x_2 = 3$. Therefore, $(3, -1, -1, -1)$ and $(-1, 3, -1, -1)$ are also the solutions from eq.(24) through eq.(27).
Similarly, if $x_1 = x_2 = -1$, thus, from eq.(26) it arrives $x_3 + x_4 = 2$; and from eq.(24) it obtains

$$x_3^2 - 2x_3 - 3 = 0 \quad (32),$$

the solutions of eq.(32) are $x_3 = 3$ or -1. If $x_3 = 3$, it is easy to get $x_4 = -1$; if $x_3 = -1$, it gets $x_4 = 3$, therefore, $(-1, -1, 3, -1)$ and $(-1, -1, -1, 3)$ are also the solutions from eq.(24) through eq.(27).
In conclusion, all the solutions of the equations are $(x_1, x_2, x_3, x_4) = (1,1,1,1)$, $(3, -1, -1, -1)$, $(-1, 3, -1, -1)$, $(-1, -1, 3, -1)$ and $(-1, -1, -1, 3)$.

6. Problem 5

Problem 5(11th IMO) [15]: Let a_1, a_2, a_3, ……a_n be real constants, and $f(x) = Cos(a_1 + x) + \frac{Cos(a_2 + x)}{2} + \frac{Cos(a_3 + x)}{2^2} ...... + \frac{Cos(a_n + x)}{2^{n-1}}$. If x_1, x_2 are real, and $f(x_1) = f(x_2) = 0$, prove that $x_2 - x_1 = m\pi$ for some integer m.

Solution: Using the mathematical induction, when n=1, according to the meaning of the problem, $f(x_1) = Cos(a_1 + x_1) = 0$, thus, $a_1 + x_1 = m_1 \pi \pm \frac{\pi}{2}$, m_1 is an integer; similarly, $f(x_2) = Cos(a_1 + x_2) = 0$, so $a_1 + x_2 = m_2 \pi \pm \frac{\pi}{2}$, m_2 is also an integer, therefore, $x_2 - x_1 = (a_1 + x_2) - (a_1 + x_1) = (m_2 - m_1)\pi = m\pi$, m is evidently an integer.
Supposing when n=k, $f_k(a_k + x_1) = f_k(a_k + x_2) = 0$, $x_2 - x_1 = m\pi$ (m is an integer) is true, thus, when n=k+1,

$$f_{k+1}(a_{k+1} + x_1) = f_k(a_k + x_1) + f(a_{k+1} + x_1) = 0 \quad (33),$$

$$f_{k+1}(a_{k+1} + x_2) = f_k(a_k + x_2) + f(a_{k+1} + x_2) = 0 \quad (34).$$

Eq.(33) and eq.(34) can be written as

$$f_{k+1}(a_{k+1} + x_1) = f_k(a_k + x_1) + \frac{Cos(a_{k+1} + x_1)}{2^k} = 0 \quad (35),$$

$$f_{k+1}(a_{k+1} + x_2) = f_k(a_k + x_2) + \frac{Cos(a_{k+1} + x_2)}{2^k} = 0 \quad (36).$$

In accord with the supposition, respectively from eq.(35) and eq.(36), it obtains $Cos(a_{k+1} + x_1) = 0$; $Cos(a_{k+1} + x_2) = 0$, therefore,

$$a_{k+1} + x_1 = p_1\pi \quad (p_1 \text{ is an integer}) \tag{37}$$;

$$a_{k+1} + x_2 = p_2\pi \quad (p_2 \text{ is an integer}) \tag{38}$$.

From eq.(38)－eq.(37), it obtains $x_2 - x_1 = (p_2 - p_1) = m\pi$ (m is evidently an integer). The problem is proved.

7. Problem 6

7 Problem 6 (the first IMO) [15]: For every integer n prove that the fraction $\frac{21n+4}{14n+3}$ cannot be reduced any further.

The solution: The fraction can be written as

$$\frac{21n+4}{14n+3} = 1 + \frac{7n+1}{14n+3} = 1 + \cfrac{1}{2 + \cfrac{1}{7n+1}} \tag{39}$$.

For all the integers of n > 0 or n = 0, or n < 0 in eq.(39), $\frac{1}{7n+1}$ evidently can't be reduced, so 2+$\frac{1}{7n+1}$ can't be reduced as well.

Thus, $\cfrac{1}{2 + \cfrac{1}{7n+1}}$ can't be reduced, therefore, eq.(39) can't be reduced any further.

8. Conclusions

With respect to the present solutions of the six problems selected from the first to 11th IMO, the paper explains a few points:

1 In order to keep the inequality eq.(8) consistent with eq.(7), the square roots of two sides of eq.(7) must take positive values, so the left side of eq.(8) should take "±" symbol for the reason of that x is a variable. Moreover, the result of eq.(9) should be substituted into eq.(2) to verify whether the result satisfies it.

2 the solution of problem 4 is not only one, besides $(x_1,x_2,x_3,x_4)=(1,1,1,1)$, there are (3,－1,－1,－1), (－1,3,－1,－1), (－1,－1,3,－1) and (－1,－1,－1,3) which also satisfy the equations from eq.(24) through eq.(27).

3 In order to understand the solution of problem 6, it is easy to prove that for the fraction $m+\frac{a}{b}$, if $\frac{a}{b}$ can be reduced, thus, $\frac{mb+a}{b}$ can be reduced; if $\frac{a}{b}$ can't be reduced, thus, $\frac{mb+a}{b}$ can't be reduced as well.

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