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New approach of mathematical studies in modeling traffic flows

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Abstract
In this study the basic model is formulated to show the effect of blocking on the dynamics of vehicles flows on the road. The dynamics of vehicles flows have been described using deterministic nonlinear dynamical system. The study divided the total number of vehicles on the road into four exclusive compartments as Free — Slow – Average – Released vehicles. The well-posedness of formulated model for vehicles flows on the road has been verified. The equilibrium points of formulated model have been identified. Further, to observe the effect of blocking in the vehicles population flows the retardation number is used. The local and Global stability of equilibrium points of the formulated model equations has been analyzed using modified Castilo Chavez approach and Hurwitz criteria. The qualitative behavior of solutions of formulated dynamical system has been described by using a MATLAB software.

Keywords: Nonlinear dynamical system, Well-posedness, Equilibrium points, Stability, Retardation number

1. Introduction

The consequences of non-managed blocking effects on vehicles flows is one of life threatening and the critical issues for both developing and developed countries [1, 3, 5]. It is well recognized that if someone gets the disease that can invade the population if not managed properly, then the person is treated separately or in safe place with high care until he/she gets recovered or her personal status improved so that she/he can manage the burden of the disease [2, 4, 6]. Also poor management of blocking of vehicles flows results in both human life and economical destruction [7, 9, 10]. Daily accident's news are heard in all over the world that caused by vehicles blocking. Thus, this study describes a mathematically model that shows how vehicles flow on the road. This study also intended to depict the effect blocking on the road and the dynamics of the vehicles population pertains to the blocking. It is commonly known that most of the deterministic models have been applied to study the dynamics of human population pertains to disease to in order to illustrate the extinction and persist of a disease entered the human population. Here, this study describes a new basic model for the stronger effect than a disease in the human population. In general, blocking is the act of barring the vehicles not to flow freely on the road. This is the strongest problem of vehicles flows on the road and needs great care transportation management involvement. It is well known that disease are treated and cured with good medication assistance given by physicians. But, most of the time the misfortune caused by this blocking does not provide time to anyone to be treated or handled. Thus needs high cares from the starting to the completion of the driving.

2. Motivation of the Study

2.1 Comparison of model of Epidemiology with model of Traffic flow

Generally, deterministic model of ordinary differential equations is used in epidemiology in order to model and analysis dynamics of human population in the presence of diseases. Now this study extended the deterministic dynamical model to model dynamics of vehicles population pertaining to blocking on the road. In the contribution of a basic nonlinear dynamical system, for modeling vehicles population flow on the road, the platform of the study is the work done in [1]. From the revision of deterministic models in epidemiology one can observe that total human population can be classified into four compartments i.e., Susceptible – Exposed – Infected – Recovered (SEIR).

1. The Susceptible compartment encloses disease free people but can get disease if exposed to it.
2. The Exposed compartment encloses people who exposed to the disease but does not propagate disease to others.
3. The Infectious compartment encloses people who infected by the disease and can propagate disease to others in allowable contact.
4. The Recovered compartment encloses people got affected by the disease recovered from the disease by some means like natural cure, treatment etc.

In similar approach to construct epidemiological deterministic models in this study the vehicles flows on a road (or drivers driving vehicles) are categorized to five compartments i.e., Free — Slow – Average – Released (FSAR).

1. The Free compartment contains unblocked vehicles bot can be blocked if they face blocking vehicles
2. The Slow compartment contains vehicles with low blocking. These vehicles can block others in allowable distance of blocking if they meet vehicles on the road.
3. The Average compartment contains middle blocked vehicles. The vehicles in this compartement are relatatively under high blocking and needs driver's attention highly relative to others compartment in order to save both the passangers in the vehicles and ressources in the vehicles or on the road.
4. The Released compartment contains blocking free vehicles after blockage.

Thus, observing that the blocking effects in traffic flow is continuously persisting with human life this study is aimed to put platform for further study of traffic flows on the road so that the advanced concepts of ODE can describe better understanding than that of PDE description of traffic flow models. In this study, in addition to what described above about vehicles flow related to blocking a different terms are used in the following sections.

3. Model Formulation

The deterministic model formulated in this study divides total vehicles population available on the road into four compartments. These are described as: Free vehicles$F(t)$, slow vehicles$S(t)$, Average vehicles$A(t)$, and Released vehicles$R(t)$. So that

$$N(t) = F(t) + S(t) + A(t) + R(t) \quad (1)$$

Here in (1), (i) $N(t)$ denotes total population size of vehicles (ii) $F(t)$denotes population size of Free Vehicles. These are vehicles that move without the influence of blocking on the road but have a chance or possibility to be blocked in the future. $S(t)$denotes the population size of slow vehicles which are partially blocked and are moving under the influence of blockings and can block others in possible allowable distance between vehicles (iv) $A\ (t)$denotes the population size of average vehicles. These are vehicles that flows under the influence of medium blocking. (v) $R\ (t)$ Denotes the population size of Released vehicles which are just released from blockings and have no chance of experiencing blockings again.

Table 1: Description of Model Variables

Variable	Description pertaining to traffic flow	Description pertaining to Epidemiology
$F(t)$	Population size of FREE vehicles	Size of Suceptible population
$S(t)$	Population size of SLOW vehicles	Size of Asymptomatic population
$A(t)$	Population size of AVERAGE vehicles	Size of Symptomatic population
$R(t)$	Population size of RELEASED vehicles	Size of Recovered population

Table 2: Description of Model Parameters

Parameter	Description pertaining to traffic flow (FSBD)	Description pertaining to Epidemiology (SEIR)
τ	Recruitment rate the rate at which new vehicles joining the road (OR) growth rate of free vehicles	Constant birth rate of susceptible population
β	Rate of free vehicles becoming Slow vehicles. That is, with this rate the free vehicles are experiencing partial blocking and are running with a reduced speed.	Transmission rate of infection to Susceptible population
ρ	Transferring rate of Slow vehicles to become Average vehicles.	Transfer rate from Exposed to Infected
γ	Transferring rate of Average vehicles to become Released vehicles	Recovery rate
ν	Natural outflow	Natural death rate

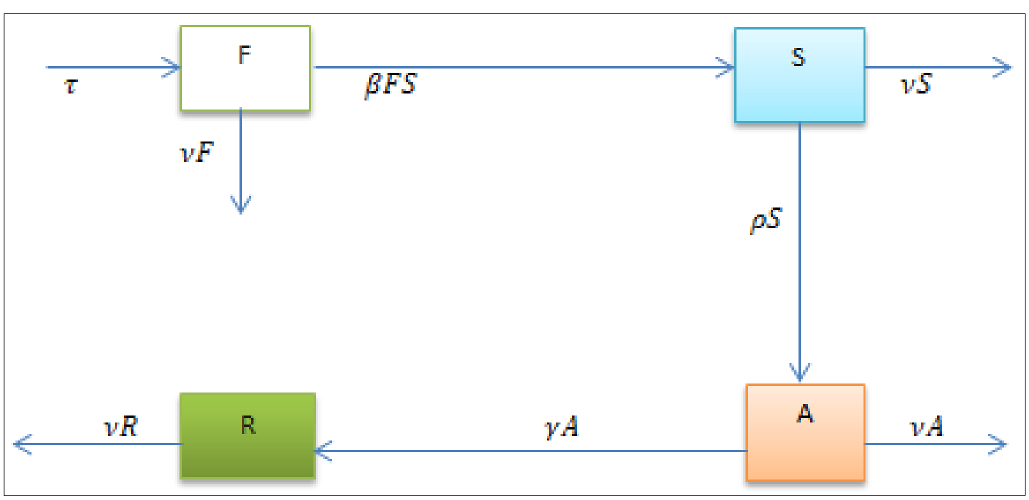

Fig 1: Progression of vehicles among Free(F), Slow(S), Average(B), and Released(R) compartments

Based on the assumptions listed above and the diagram given in Figure 2, the mathematical model describing the dynamics of population sizes vehicles subject to blocking on a road can be expressed as a system of nonlinear differential equations as

$$dF/dt = \tau - \beta FS - \nu F \quad (2)$$

$$dS/dt = \beta FS - \rho S - \nu S \quad (3)$$

$$dA/dt = \rho S - \gamma A - \nu A \quad (4)$$

$$dR/dt = \gamma A - \nu R \quad (5)$$

With initial conditions, $F(0), S(0), A(0)$ and $R(0)$.

4. Analysis of the Model

In this section mathematical analysis of the model (2) – (5) is carried out. The analysis comprises of the following features: (i) Existence, positivity and boundedness of solutions (ii) Equilibrium points (iii) Blocking Free equilibrium points (iv)Endemic equilibrium points (v) Basic retardation number (vi) Stability analysis of the blocking free equilibrium points (vii) Local stability of blocking free equilibrium point (viii) Global stability of blocking free equilibrium point. These mathematical aspects are presented and explained in the following sub-sections respectively.

4.1 Existence, Positivity and Boundedness of solution

In order that the model equations (2) – (5) have physically valid meaning for the modeled problem of traffic flow and mathematically well posed the following theorems are stated and proved. [See, 12, 13]

4.1.1 Positivity of the solutions

Theorem 1: If the initial conditions $F(0), S(0), A(0)$ and $R(0)$ are non-negative then the solution region $R = \{F(t), \quad S(t), \quad A(t), \quad R(t)\}$ of the system of equations (2) – (5) is non-negative.

Proof: To show that the solution of equations (2) – (5) is non-negative here each model equation of the dynamical system is considered separately and shown that it has a non-negative solution as follows:

Positivity of $F(t)$*:* Consider the model equation (2) given by $dF/dt = \tau - \beta FS - \nu F$ which without loss of generality, after discarding the positive term τ, (2) can be expressed as an inequality as $dF/dt \geq -(\beta S + \nu)F$. This differential inequality, being first order and linear, can be solved easily to find its solution as $F(t) \geq F(0)\ Exp\ \left\{-\int_0^t [\beta S + \nu] dt\right\}$. Here the integral constant $F(0)$ represents initial population size and by definition it is a positive quantity. Also, as per the definition of exponential functions, the exponential factor $Exp\ \left\{-\int_0^t [\beta S + \nu] dt\right\}$ is always non-negative for any value of the exponent. Hence, it can be concluded that $F(t)$ is a non-negative quantity i.e., $F(t) \geq 0$.

Positivity of $S(t)$*:* Consider the model equation (3) given by $dS/dt = \beta FS - \rho S - \nu S$ which without loss of generality, after discarding the positive term βFS, can be expressed as an inequality as $dS/dt \geq -(\rho + \nu)S$. This differential inequality, being first order and linear, can be solved easily to find its solution as $S(t) \geq S(0)\ Exp\ \{-(\rho + \nu)t\}$. Here the integral constant $S(0)$ represents initial population size and by definition it is a positive quantity. Also, as per the definition of exponential functions, the exponential factor $Exp\ \{-(\rho + \nu)t\}$ is always non-negative for any value of the exponent. Hence, it can be concluded that $S(t)$ is a non-negative quantity i.e., $S(t) \geq 0$ for all $t \in [0, \infty)$.

Positivity of $A(t)$*:* Consider the model equation (4) given by $dA/dt = \rho S - \gamma A - \nu A$ which without loss of generality, after discarding the positive term ρS, can be expressed as an inequality as $dA/dt \geq -(\gamma + \nu)A$. This differential inequality, being first order and linear, can be solved easily to find its solution as $A(t) \geq A(0)\ Exp\ \{-(\gamma + \nu)t\}$. Here the integral constant $A(0)$ represents initial population size and by definition it is a positive quantity. Also, as per the definition of exponential functions, the exponential factor $Exp\ \{-(\gamma + \nu)t\}$ is always non-negative for any value of the exponent. Hence, it can be concluded that $A(t)$ is a non-negative quantity i.e., $A(t) \geq 0$ for all $t \in [0, \infty)$.

Positivity of $R(t)$*:* Consider the model equation (6) given by $dR/dt = \gamma A - \nu R$ which without loss of generality, after discarding the positive term γA can be expressed as an inequality as $dR/dt \geq -\nu R$. This differential inequality, being first order and linear, can be solved easily to find its solution as $R(t) \geq R(0)\ Exp\ \{-\nu t\}$. Here the integral constant $R(0)$ represents initial population size and by definition it is a positive quantity. Also, as per the definition of exponential functions, the exponential factor $Exp\ \{-\nu t\}$ is always non-negative for any value of the exponent. Hence, it can be concluded that $R(t)$ is a non-negative quantity i.e., $R(t) \geq 0$ for all $t \in [0, \infty)$.

Since, obviously exponential expressions are positive and initial conditions are non-negative it can be concluded that the solutions region D is a set containing non-negative quantities. Thus,

$D = \{[F(t), \quad S(t), \quad A(t), \quad R(t)] \ni F(t) \geq 0, S(t) \geq 0, A(t) \geq 0, R(t) \geq 0, \forall t \in [0, \infty)\}$.

4.1.2 Boundedness of the solutions region

In order to make the formulated model is valid and well posed it is also necessary to show that the solutions region is bounded. This fact has been stated as Theorem 2 and verified in its proof following [1].

Theorem 2 The non-negative solutions region
$R = \{F(t), \quad S(t), \quad A(t), \quad R(t)\}$ Of the system of equations (2) – (5) is bounded i.e., $N(t) \leq \tau/\nu$.

Proof:
To show that the solutions region of the system of equations (2) – (5) is bounded, it is sufficient if it is shown that the total population is bounded. Now adding all terms on the right and left sides of equations in dynamical system (2) – (5) gives the resultant equation as
$(dF/dt) + (dS/dt) + (dA/dt) + (dR/dt)$
$= \tau - \beta FS - \nu F + \beta FS - \rho S - \nu S + \rho S - \gamma A - \nu A + \gamma A - \nu R$

Now, on the left hand side of the above equation, some pairs of terms with same expressions but with opposite signs can be found. Hence, the equation can be expressed in simplified form as

$dN/dt = \tau - \nu N$

Now, for the first order linear equation with constant coefficients $dN/dt = \tau - \nu N$ it is straight forward to find the complete solution as$N(t) = (\tau/\nu) - [(\tau/\nu) - N(0)]e^{-\nu t}$. Here $N(0)$ is the initial size of all categories of vehicles on the road. It follows that $N(t)$ is bounded as $t \rightarrow \infty$ i.e., $N(t) \leq (\tau/\nu)$ provided that the condition $N(0) \leq (\tau/\nu)$is satisfied. Thus, it can be concluded that the solutions region $D = \{F(t), \quad S(t) \quad A(t), \quad R(t)\}$ is bounded i.e., $N(t) \leq (\tau/\nu)$.

4.1.3 Existence and Uniqueness of the solutions

Here it is to show that a solution for the system (2) – (5) exists and is unique following the procedure given in Derric and Grossman 1976 the solution exists and is unique. The existence and uniqueness of the solution can be stated as shown in Theorem 3.
Theorem 3: Consider a system of n first order differential equations of the type $x_i' = f_i(x_1, x_2, x_3 \dots, x_n, t)$ together with the initial conditions $x_i(t_0) = x_{i0}$where$i = 1, \dots, n$. Let D denotes a region in $(n+1)$-dimensional space among which one dimension is for t and n dimensions are for the vector$x$. If all the partial derivatives $\partial f_i/\partial x_j$ for all $i, j = 1,2, \dots n$ are continuous and bounded in $D = \{(x,t), |t - t_0| \leq a, |x - x_0| \leq b\}$ then there is exists a constant $\delta > 0$such that there an unique continuous vector solution$x^* = [x_1(t), x_2(t), \dots, x_n(t)]$ in the interval $|t - t_0| \leq \delta$ for the system of n equations. Now accordingly let state the theorem and prove (2) – (5). That is,

$$dF/dt = \tau - \beta FS - \nu F \quad (2)$$

$$dS/dt = \beta FS - \rho S - \nu S \quad (3)$$

$$dA/dt = \rho S - \gamma A - \nu A \quad (4)$$

$$dR/dt = \gamma A - \nu R \quad (5)$$

Theorem 4: There exist an unique solution to the system of equations (2) – (5).
Proof: The statement here is proved following the procedure given in Theorem 3. Now, the system of equations (2) – (5) together with the initial conditions can be expressed as
$dF/dt = \tau - \beta FS - \nu F \equiv f_1, F(t_0) = F_0$
$dS/dt = \beta FS - \rho S - \nu S \equiv f_2, S(t_0) = S_0$
$dA/dt = \rho S - \gamma A - \nu A \equiv f_3, A(t_0) = A_0$
$dR/dt = \gamma A - \nu R \equiv f_4, R(t_0) = R_0$

To show existence and boundedness let us show that partial derivatives of functions f_i with respect to variables be continuous and bounded. Where $i = 1, \quad 2, \quad 3, \quad 4,$

Table 3: Showing continuity and bounded ness of the partial derivatives

Function	Continuity	Bounded ness
f_1	$\partial f_1/\partial F = -\beta S - \nu$ $\partial f_1/\partial S = -\beta F$	$\lvert\partial f_1/\partial F\rvert = \lvert-\beta S - \nu\rvert < \infty$ $\lvert\partial f_1/\partial S\rvert = \beta F < \infty$

	$\partial f_1/\partial A = 0$ $\partial f_1/\partial R = 0$	$\lvert\partial f_1/\partial A\rvert = 0 < \infty$ $\lvert\partial f_1/\partial R\rvert = 0 < \infty$
f_2	$\partial f_2/\partial F = \beta S$ $\partial f_2/\partial S = \beta F - \rho - \nu$ $\partial f_2/\partial A = 0$ $\partial f_2/\partial R = 0$	$\lvert\partial f_2/\partial F\rvert = \lvert\beta S\rvert < \infty$ $\lvert\partial f_2/\partial S\rvert = \lvert\beta F - \rho - \nu\rvert < \infty$ $\lvert\partial f_2/\partial A\rvert = 0 < \infty$ $\lvert\partial f_2/\partial R\rvert = 0 < \infty$
f_3	$\partial f_3/\partial F = 0$ $\partial f_3/\partial S = -\eta - \nu$ $\partial f_3/\partial A = -\gamma - \nu$ $\partial f_3/\partial R = 0$	$\lvert\partial f_3/\partial F\rvert = 0 < \infty$ $\lvert\partial f_3/\partial S\rvert = \eta + \nu < \infty$ $\lvert\partial f_3/\partial A\rvert = \gamma + \nu < \infty$ $\lvert\partial f_3/\partial R\rvert = 0 < \infty$
f_4	$\partial f_4/\partial F = 0$ $\partial f_4/\partial S = 0$ $\partial f_4/\partial A = \gamma$ $\partial f_4/\partial R = 0$	$\lvert\partial f_4/\partial F\rvert = 0 < \infty$ $\lvert\partial f_4/\partial S\rvert = 0 < \infty$ $\lvert\partial f_4/\partial A\rvert = \gamma < \infty$ $\lvert\partial f_4/\partial R\rvert = 0 < \infty$

Hence, by Derric and Grossman 1976 the solution exists and is unique.

4.2 Equilibrium points

In order to have a better understanding about the dynamics of a model, the equilibrium points of the solution region are to be identified and their stability analysis is to be conducted. In this section such identification and analysis is conducted.

An equilibrium solution is a steady state solution of the model equations (2) – (5) in the sense that if the system begins at such a state, it will remain there for all times as long as any disturbance occurs. In other words, the population sizes remain unchanged and thus the rate of change for each population vanishes. Equilibrium points of the model are found, categorized, stability analysis is conducted and the results have been presented in the following:

3.2.1 Blocking free equilibrium BFE

At blocking free equilibrium vehicles flow freely without any interference of any kind of blocking. That is, at this equilibrium vehicles will run freely with speeds as per the wish of their drivers. Furthermore, at this equilibrium no vehicle is forced either to run with slower speeds or to stop completely. That is, $S = A = 0.$ thus, under this assumption the solutions of system of equations (2) – (5) is given by:

$$F = \tau/\nu\,, S = 0, A = 0, R = 0$$

Thus, blocking free equilibrium BFE of the model is obtained as

$$E_0 = (\tau/\nu\,, \quad 0, \quad 0, \quad 0)$$

4.2.2 The endemic equilibrium point

Let $E_1 = (F^1, \quad S^1, \quad A^1, \quad R^1)$ be an endemic equilibrium point. In order to obtain endemic equilibrium E_1 point of the model the left hand sides of the equations (2) – (6) are set equal to zero. Thus, the model equations reduce to the form as

$$\tau - \beta FS - \nu F = 0 \tag{6}$$

$$\beta FS - \rho S - \nu S = 0 \tag{7}$$

$$\rho S - \gamma A - \nu A = 0 \tag{8}$$

$$\gamma A - \nu R = 0 \tag{9}$$

The endemic equilibrium is the solution of the set of equations (6) – (9). On employing simple algebraic manipulations the solution can be obtained as

$F^1 = \tau/(\nu R_0)\,, S^1 = (\nu/\beta)(R_0 - 1), A^1 = [\rho\nu\ /\ \beta(\gamma + \upsilon)\ ](R_0 - 1)$

$R^1 = [\rho\nu\gamma\ /\ \nu\beta(\gamma + \upsilon)\ ](R_0 - 1)$

Hence, the endemic equilibrium point of the model is given by

$E_1 = (F^1, \quad S^1, \quad A^1, \quad R^1)$

4.3 Derivation of basic Retardation number ($\boldsymbol{R_0}$)

Basic retardation number has similar meaning to that of basic reproduction number in Epidemiology and it is described as the average number of blocked vehicles generated by one blocked vehicle when it is introduced into fully freely flowing vehicles. Calculating retardation number is important to analyze the local stability of nonlinear system of equations (2) – (5).

The retardation number is the largest eigenvalue of the matrix$K = \mathcal{F}V^{-1}$. Where,

$$\mathcal{F} = \left(\partial f/\partial x_j\right)|_{E_0}, V = \left(\partial v/\partial x_j\right)|_{E_0} \tag{7}$$

Here, f is the newly blocking terms and v is non-singular matrix of the remaining transfer terms. Now, the basic retardation number R_0 of the model (2) – (5) is computed using the next generation matrix in similar procedure as reproduction number in epidemiological concept used to be computed. Thus, the next generation matrices $f, v, \mathcal{F}, V, V^{-1}$, $\mathcal{F}\,V^{-1}$ are constructed respectively as follows:

$f = \begin{bmatrix} \beta FS \\ 0 \end{bmatrix}$ Implies $\mathcal{F} = \begin{bmatrix} \beta\tau/\nu & 0 \\ 0 & 0 \end{bmatrix}$

$v = \begin{bmatrix} (\rho+\nu)S \\ -\rho S + (\gamma+\nu)A \end{bmatrix}$ Implies $V = \begin{bmatrix} \rho+\nu & 0 \\ -\rho & \gamma+v \end{bmatrix}$

Now, $K = \mathcal{F}\,V^{-1} = \begin{bmatrix} (\beta\tau)/\nu(\rho+\nu) & 0 \\ 0 & 0 \end{bmatrix}$

Clearly, eigenvalues of the next generation matrix K are
$\lambda_1 = (\beta\tau)/\nu(\rho+\nu)$ & $\lambda_2 = 0$ Among the eigenvalues λ_1 is a larger. Hence, the Retardation number of the model is given by
$R_0 = \rho(\mathcal{F}V^{-1}) = (\beta\tau)/\nu(\rho+\nu)$

4.4 Stability analysis of the blocking free equilibrium

In the absence of blocking, the traffic flow model will have a unique blocking free steady state E_0. To find the local stability of E_0, the Jacobian matrix of the model equations valued at blocking free equilibrium point E_0 is used. It is already shown that the BFE of model (2) – (5) is given by $E_0 = (\tau/\nu, \quad 0, \quad 0, \quad 0)$. Now, the stability analysis of BFE is conducted and the results are presented in the form of theorems and proofs in the following.

3.4.1 Local stability of blocking free equilibrium

Let J be the Jacobian matrix formed from system of equations (2) – (5). Thus, following the procedures given in the literature [1] local stability of blocking free equilibrium point is found.

Theorem 1. Let E_0 is a BFE of the system of equations and all eigenvalues of Jacobian matrix at E_0 are negative, then E_0 is locally asymptotically stable if$R_0 = \rho(FV^{-1}) < 1$, but unstable if$R_0 > 1$.

Proof: To prove this theorem first the Jacobian matrix is constructed from the model equations (2) – (5). That is,

$$J(F, \quad S, \quad A, \quad R) = \begin{bmatrix} -\beta F - \nu & -\beta F & 0 & 0 \\ \beta S & \beta F - \rho - \nu & 0 & 0 \\ 0 & \rho & -\gamma - \nu & 0 \\ 0 & 0 & \gamma & -\nu \end{bmatrix}$$

Now, the Jacobian matrix J reduces to the form, at the blocking free equilibrium point, as

$$J(\tau/\nu, \quad 0, \quad 0, \quad 0, \quad 0, \quad 0) = \begin{bmatrix} -R_0(\rho+\nu) - \nu & -\beta\tau/\nu & 0 & 0 \\ 0 & (\rho+\nu)(R_0-1) & 0 & 0 \\ 0 & \rho & -\gamma-\nu & 0 \\ 0 & 0 & \gamma & -\nu \end{bmatrix}$$

Now, the eigenvalues of Jacobian matrix at blocking free equilibrium is obtained after solving the characteristic equation$det(J(E_0) - \lambda I) = 0$. Thus the eigenvalues are:
$\lambda_1 = -\nu, \lambda_2 = -\nu - \gamma, \lambda_3 = -\nu - R_0\rho - R_0\nu, \lambda_{4,} = (\rho+\nu)(R_0-1)$. Here, it is observable that all eigenvalues are negative provided that $R_0 < 1$.

Thus, from Hurwitz criteria it can be concluded that blocking free equilibrium point is locally asymptotically stable for $R_0 < 1$ provided that the above conditions are satisfied.

Theorem 5 If $(V - \mathcal{F})$ is non-singular M-matrix and $(F_0 - F)B \geq 0$, then the blocking free equilibrium is globally asymptotically stable for $\rho(\mathcal{F}V^{-1}) < 1$ [1, 8, 11].

Proof: From the computation done above it is observable that $\mathcal{F}$ is non-negative and V is non-singular M-matrix. Further it is clear that $(V - \mathcal{F})$ is non-singular M-matrix for$\rho(\mathcal{F}V^{-1}) < 1$. Now, to show that the blocking-free equilibrium is globally asymptotically stable for$R_0 < 1$, it is sufficient to show that F $\leq F_0$. From the total population $N(t)$ we have, $N(t) = F(t) + S(t) + A(t) + R(t)$which satisfies $N'(t) = \tau - \nu N(t)$, so that $N(t) = F_0 - (F_0 - N(0))e^{-\nu t}$, with $F_0 = \tau/\nu$. If$N(0) \leq F_0$, then $F(t) \leq N(t) \leq F_0$ for all time. If, on the other hand,$N(0) > F_0$, then N(t) decays exponentially to F_0, and either $F(t) \rightarrow F_0$, or there is some time T after which $F(t) < F_0$. Thus, the blocking free equilibrium is globally asymptotically stable for $\rho(\mathcal{F}V^{-1}) < 1$.

5. Numerical Simulations

The numerical simulation study of nonlinear dynamical system has been carried-out using MATLAB. The following numerical values are assigned to parameters and variables and are used to describe the blocking effects on flow of vehicles. This simulation study describes the blocking effect on the motions of vehicles of all categories on a road. The simulated vehicles flows have been observed with different time intervals.

Table 4: List of values assigned to the parameters of the model

Parameter	Value	Source
τ	40	[1]
β	0.0004	Assumed
ν	0.02	Assumed
ρ	0.095	Assumed
γ	0.08	Assumed

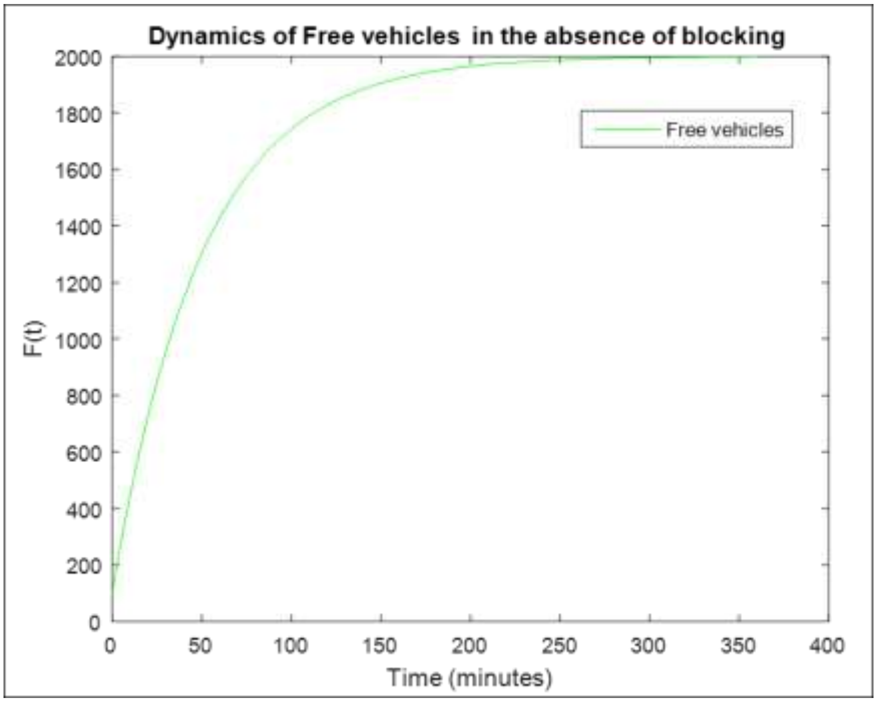

Fig 1: Free vehicles dynamics on the road with retardation greater than one

In the Figure 1 the free flowing vehicles are increasing from the beginning until they arrive at the upper bound of total vehicles and finally continue flowing with the same inflow and natural outflow.

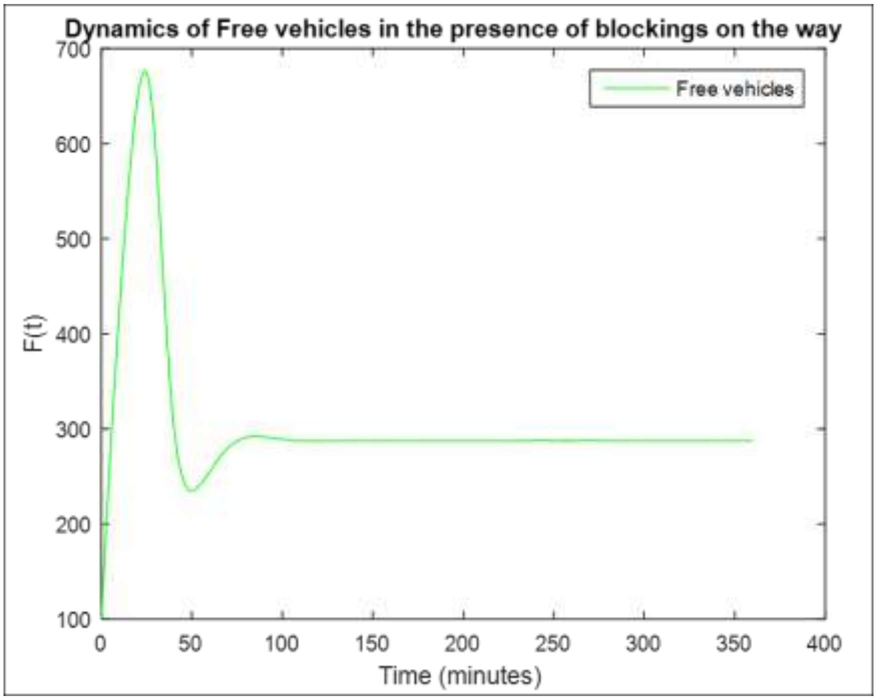

Fig 2: Free vehicles dynamics in the presence of blocking

Figure 2 shows that initially the free vehicles increased and after some minutes decreased because of blocking and continue with unrecognizable increasing and decreasing on the road.

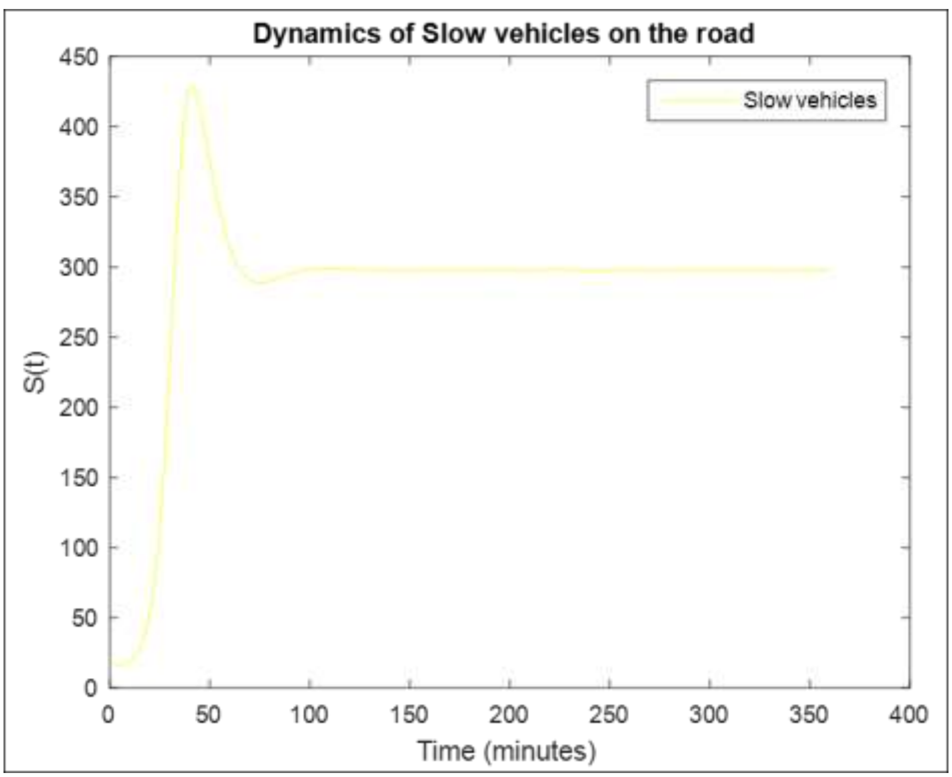

Fig 3: Dynamics of Slow vehicles with retardation number greater than one

Figure 3 describes the dynamics of slow vehicles on the road. This simulation shows that initially the number of slow vehicles decrease and followed with strictly increase upto aroud fifty minutes then decrease followed with unrecognizable changes in the numbers as time increases.

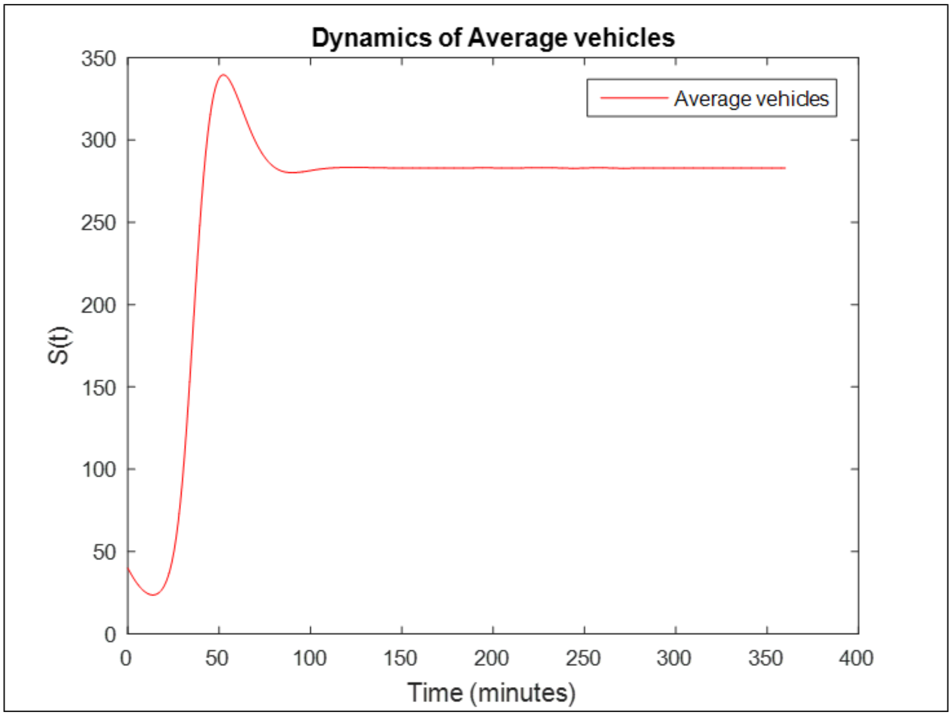

Fig 4: Dynamics of Average vehicles with retardation number greater than one

Figure 4 shows the dynamics of slow vehicles on the road. This simulation shows that initially the number of slow vehicles decrease and followed with strictly increase upto some minutes then decrease followed with unrecognizable changes in the numbers as time increases.

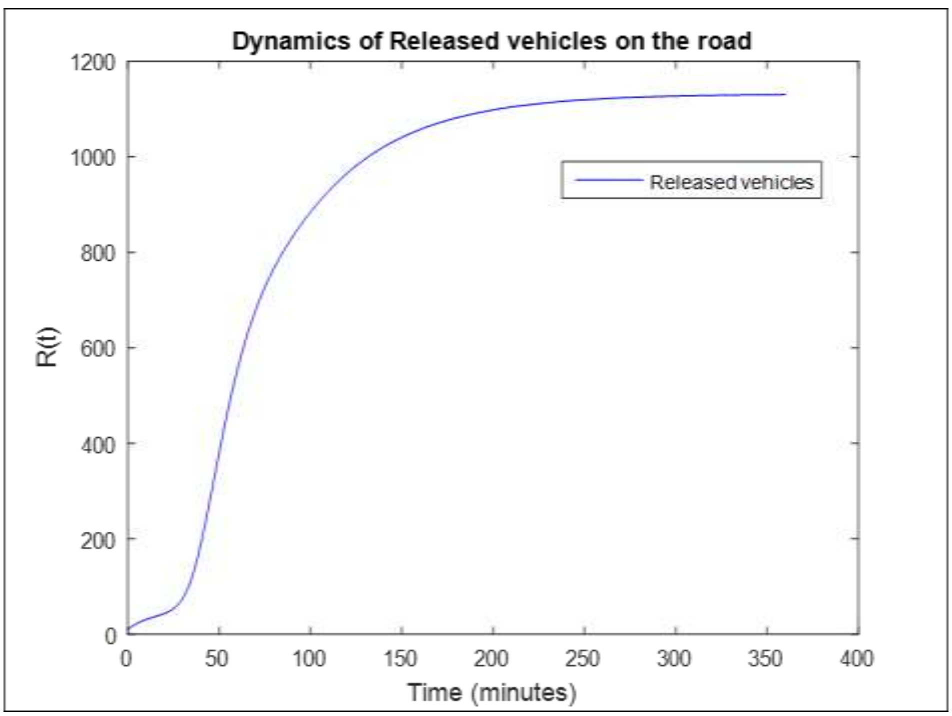

Fig 5: Dynamics of Released vehicles with retardation number greater than one

Figure 5 describes the dynamics of Released vehicles on the road. This simulation shows that initially the number of Released veicles increase upto some tme followed with unrecognizable changes in the numbers as time increases.

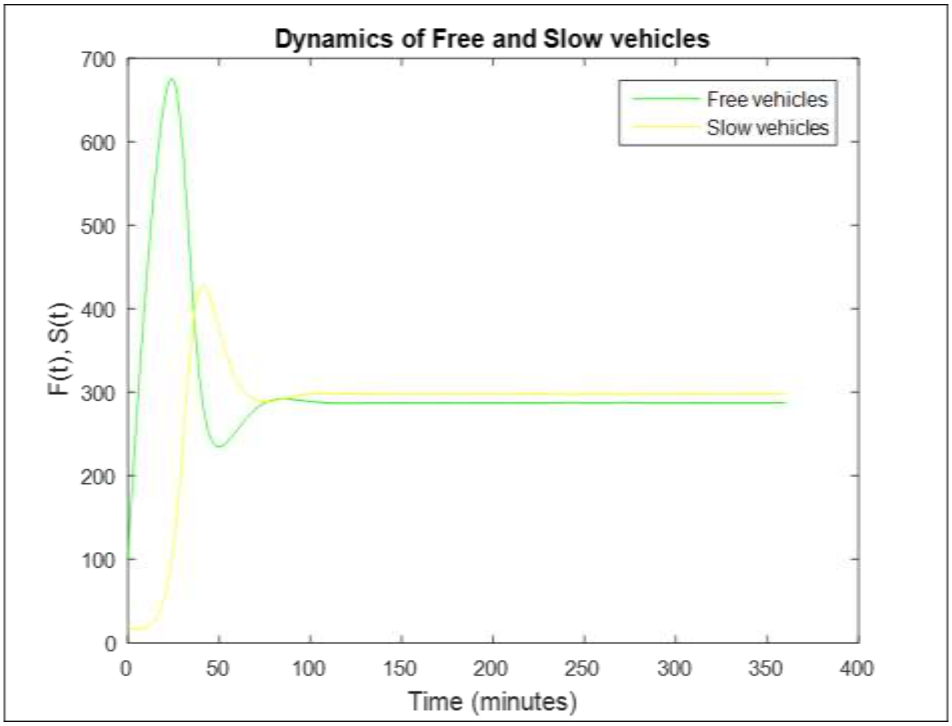

Fig 6: Free and Slow vehicles with retardation number greater than one

Figure 6 describes the dynamics of free vehicles and slow vehicles on the road. This simulation shows that initially the number of free vehicles increase but slow vehicles decrease. However, both increase and decrease on some interval followed with both opposite decrements and increament up to sometime followed at which no changes in the compartments.

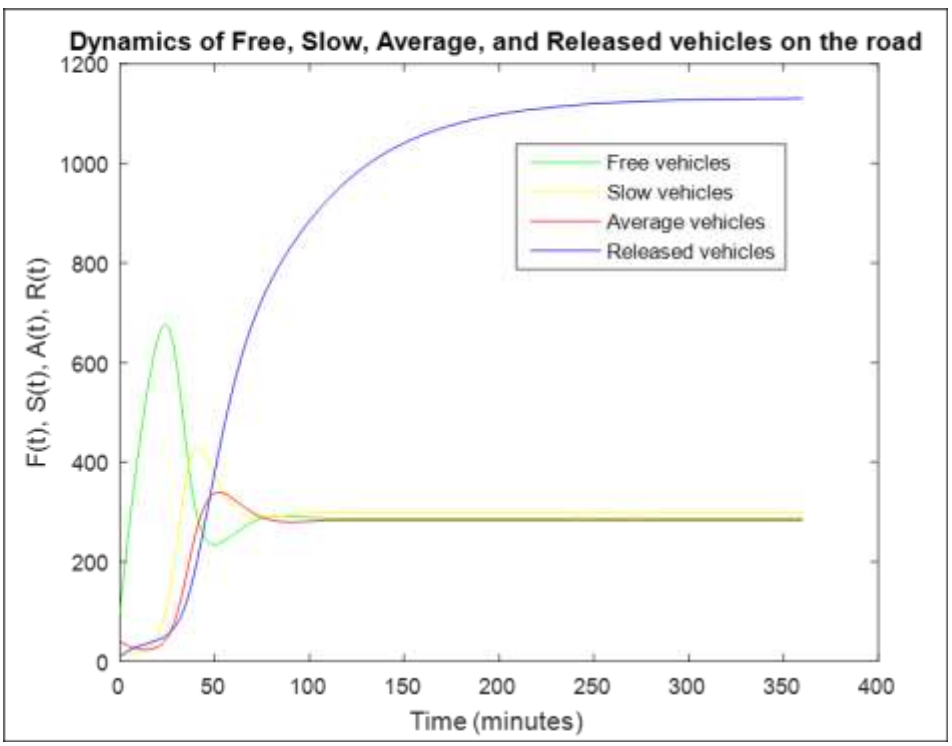

Fig 7: Free, Exposed, Slow, Average and Released vehicles dynamics

Figure 7 describes the dynamics of the total vehicles on the road. This simulation shows that as the number of free vehicles on the road increases then the number of vehicles under blocking gets decreasing but as the number of free vehicles decrease the number of vehicles in blocking compartment gets increase. Finally, after some time all meet the point where there is no change in numbers of vehicles in each compartment as time increases. However, the Released vehicles increase up to sometime followed with constant number as time increase.

5. Results and Discussion

From the this study supported with MATLAB simulation the blocking has high effect on the flow of the vehicles. The more blocked vehicles available on the road the more time usage it takes for the passangengers and drivers on the road. Less blocked vehicles the more free flowing vehicles on the road. The transmission and recruitment rates are the most sensitive parameters that contribute to the increament of the blockage.

6. Conclusion

In this study, basic model of vehicles flows has been formulated. The existence, positivity and boundedness of the formulated model is verified to illustrate that the model is physically meaningful and mathematically well posed. In particular, the stability analyses of the model were investigated using the basic reproduction number and Routh Hurwitz criterion. Blocking reduces number of freely flowing vehicles. We observed that vehicles flows behaves like harmonic oscillation motion.

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8. References

1. Kumama Regassa, Purnachandra Rao Koya. Modeling and Analysis of Traffic Flows using Four Dimensional Non-linear Dynamical System of Ordinary Differential Equation. IOSR Journal of Mathematics (IOSR-JM), e-ISSN: 2278-5728, p-ISSN: 2319-765X. 2019; 15(4) 08-24. Ser. I PP www.iosrjournals.org
2. Van-den-Driessche P, Watmough J. Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission. Math. Biosci. 2002; 180:29-48.
3. Caramia M, D'Apice C, Piccoli B, Sgalambro A. Fluidsim: A Car Traffic Simulation Prototype Based on Fluid Dynamic, Algorithms. 2010; 3:294-310.
4. Diekmann O, Heesterbeek JAP. Mathematical epidemiology of infectious diseases, Wiley Series in Mathematical and Computational Biology, Wiley, Chichester, 2000.
5. Chandler RE, Herman R, Montroll EW. Traffic Dynamics: Studies in Car Following, Operations Research. 1958; 6:165-184.
6. Brauer F, Vanden Driessche P, Wu J. (Eds.). Mathematical Epidemiology, Lecture Notes in Mathematics, Mathematical Biosciences subseries 1945, Springer, Berlin - Heidelberg - New York, 2008.
7. Richards PI. Shock Waves on the Highway, Operations Research. 1956; 4:42-51.
8. Castillo-Chavez C, Feng Z, Huang W. On the computation R0 its role on global stability, Mathematical approaches for emerging and re-emerging infectious diseases. IMA, 2002; 125:229-250.
9. Lighthill MJ, Whitham GB. On Kinematic Waves. I: Flood Movement in Long Rivers, On Kinematic Waves. II: A Theory of Traffic Flow on Long Crowded Roads, Proc. Royal Soc. A, 1955; 229:281-345.
10. Bando M, Hasebe K, Nakayama A, Shibata A, Sugiyama Y. Dynamical Model of Traffic Congestion and Numerical Simulation, Phys. Rev. E. 1995; 51:1035-1042.
11. Kumama Regassa, Purnachandra Rao Koya. Modeling and Analysis of Population Dynamics of Human Cells Pertaining to HIV/AIDS with Treatment. American Journal of Applied Mathematics. 2019; 7(4):127-136. doi: 10.11648/j.ajam.20190704.14
12. Kumama Regassa Cheneke, Geremew Kenassa Edessa, Purnachandra Rao Koya. Global Stability and Sensitivity Analysis of the Dynamics of Human Population Subjected to HIV/AIDS with Treatment. IOSR Journal of Mathematics (IOSR-JM) e-ISSN: 2278-5728, p-ISSN: 2319-765X. 2019; 15(6):34-51. Ser. III (Nov – Dec 2019).
13. Kumama Regassa Cheneke, Geremew Kenassa Edessa, Purnachandra Rao Koya. Impact of Treatment and Isolation on the Dynamics of HIV Transmission. IOSR Journal of Mathematics (IOSR-JM), e-ISSN: 2278-5728, p-ISSN: 2319-765X. 2019; 15(6):73-72. Ser. V. DOI: 10.9790/5728-1506055372