International Journal of Physics and Mathematics
www.physicsjournal.net
Online ISSN: 2664-8644; Print ISSN: 2664-8636
Received: 03-11-2018; Accepted: 04-12-2018; Published: 04-01-2019
Volume 1; Issue 1; 2019; Page No. 09-13

The differential transform method solution for nonlinear duffing equation involving both integral and non-integral forcing terms

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Abstract
In this paper, the differential transform method is used for solving nonlinear Duffing equation involving both integral and non-integral forcing terms. The differential transform of nonlinear terms are computed using Adomian polynomials. The proposed approach reduces the considered problem to a single nonlinear algebraic equation in one unknown parameter. The numerical examples illustrate the simplicity of the proposed technique and show good agreement between the obtained and the exact solutions.

Keywords: Duffing equation; integral forcing terms; differential transform method; Adomian polynomials

1. Introduction

Consider the following Duffing equation involving both integral and non-integral forcing terms

$$u''(x) + \sigma u'(x) + f(x,u,u') + \int_0^x k\big(x,s,u(s)\big)ds = 0, 0 < x < 1, \tag{1}$$

Subject to the separated boundary conditions

$$\begin{aligned} p_0 u(0) - q_0 u'(0) &= a, \\ p_1 u(1) + q_1 u'(1) &= b, \end{aligned} \tag{2}$$

Where $f: [0,1] \times \mathbb{R}^2 \to \mathbb{R}, k: [0,1] \times 0,1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions, $\sigma \in \mathbb{R} - \{0\}$, and p_0, p_1, q_0, q_1, a, b are real positive numbers. The Duffing equation is used as a mathematical model that describes a classical oscillator in a double-well by a periodical driven, which gains a lot of attention in the study of several chaotic phenomena [1] and [2]. It is also utilized in the study of different scientific applications which include extraction of periodic orbits, infinite-domain nonuniformity, nonlinear mechanical oscillators, and prediction of diseases [3, 7]. The existence and uniqueness of solution of problem (1)-(2) is presented in [8]. The literature of numerical techniques on the solution of problem (1)-(2) contains for example using a pseudospectral method in [9], an improved variational iteration method in [10], and an iterative reproducing kernel method in [11]. In this work, we propose an approach to solve the nonlinear Duffing equation (1)-(2) using the differential transform method (DTM) with Adomian polynomials and illustrate the advantages of this approach.

The DTM [12] is an iterative procedure for obtaining Taylor series solution of functional equations. The main advantage of this method is that it directly transforms a given problem into an algebraic recurrence formula without requiring linearization, discretization or perturbation. This reduces the size of computational work while providing as many solution terms as needed. The DTM has also been utilized to solve boundary value problems as presented in [13] and [14]. The DTM was used for solving nonlinear problems as well, but only when the nonlinear terms are presented in the form of polynomials and products of unknown function with its derivatives. Yet recently an algorithm has been designed for using Adomian polynomials to compute the differential transform of analytic nonlinear terms of any form. This algorithm yields good results for ordinary [15], partial [16], and fractional oder differential equations [17].

In this article, we use the DTM with Adomian polynomials to solve nonlinear Duffing equation that involves forcing terms of both integral and non-integral forms when subjected to separated boundary conditions as represented by equations (1)-(2). In section 2, we illustrate how the proposed approach reduces the considered problem to a problem of solving a nonlinear algebraic equation. Two examples are presented in section 3 to validate the suggested approach. Section 4 contains the conclusion of this work.

2. The solution technique

2.1 The DTM

The differential transformation of the of analytic function $u(x)$ is defined by

$$U(k) = \frac{1}{k!}\Big[\frac{d^k u(x)}{dx^k}\Big]_{x=x_0}, k = 1,2,\ldots, \tag{3}$$

And the series solution in terms of $U(k)$ is defined by

$$u(x) = \sum_{k=0}^{\infty} U(k)(x - x_0)^k. \tag{4}$$

In computations, function $u(x)$ is substituted by the truncated finite series of the form

$$u(x) = \sum_{k=0}^{N} U(k)(x - x_0)^k. \tag{5}$$

Some basic properties of the differential transformation are as follows [12]. Let$u(x)$, $v(x)$ and $w(x)$ be three uncorrelated functions of variable x and let$U(k)$,$V(k)$, and $W(k)$ be their corresponding differential transformations. Then:

1. If $u(x) = v(x) \pm w(x)$, then$U(k) = V(k) \pm W(k)$.
2. If $u(x) = av(x)$, then $U(k) = aV(k)$, where a is a constant?
3. If $u(x) = v(x)w(x)$, then $U(k) = \sum_{k_i=0}^{k} V(k_i)W(k - k_i)$.
4. If $u(x) = x^n$, then $U(k) = \delta(k - n)$ where $\delta(k - n) = \begin{cases} 1 & k = n \\ 0 & k \neq n \end{cases}$.
5. If $u(x) = \frac{d^n}{dx^n} v(x)$, then $U(k) = \frac{(k+n)!}{k!} V(k + n)$.
6. If $u(x) = \int_0^x v(t)dt$, then $U(k) = \frac{V(k-1)}{k}$.

2.2 DTM with Adomian polynomials

To illustrate how the Adomian polynomials are utilized with the DTM, consider a nonlinear function $f(v)$. Then, the Adomian polynomials approximating $f(v)$ can be arranged in the form

$$\begin{aligned} A_0 &= f(v_0) \\ A_1 &= v_1 f^{(1)}(v_0) \\ A_2 &= v_2 f^{(1)}(v_0) + \frac{1}{2!} v_1^2 f^{(2)}(v_0) \\ A_3 &= v_3 f^{(1)}(v_0) + v_1 v_2 f^{(2)}(v_0) + \frac{1}{3!} v_1^3 f^{(3)}(v_0) \\ &\vdots \end{aligned} \tag{6}$$

Then the following theorem holds.

Theorem 1 *Let $f(v(x))$ be a nonlinear function where both functions $f(v)$ and $v(x)$ are differentiable to order k at the neighborhood of a point(x_0). Then, the differential transform $F(k)$ of $f(v(x))$ is obtained by replacing each v_k and $\frac{d^r v_k}{dx^r}$ in the polynomials A_k defined by (6) by $V(k)$ and$\frac{(k+r)!}{k!} V(k + r)$, respectively.*

Proof. This is the single-variable case of the proof in [16].

2.3 The proposed approach

Consider the Duffing equation in the form of equations (1)-(2). First, we set up some unknown parameters to play the role as assumed initial conditions for the considered problem as

$$u(0) = c_0, u'(0) = c_1 \tag{7}$$

Then by substituting the series solution in the first equation in boundary conditions (2), we have

$$\frac{p_0 c_0 - a}{q_0} = c_1. \tag{8}$$

Then the conditions take the following form

$$u(0) = c_0, u'(0) = \frac{p_0 c_0 - a}{q_0}. \tag{9}$$

The DTM rules are applied to the terms of problem (1) with conditions (9) and an algebraic recurrence relation is composed for $k = 0,1,2, \ldots$ which takes the form

$$\begin{cases} Y(k+2) = \frac{1}{(k+1)(k+2)}(-\sigma(k+1)Y(k+1) - F(k) - I(k)), \\ Y(0) = c_0, Y(1) = \frac{p_0 c_0 - a}{q_0}, \end{cases} \tag{10}$$

Where $F(k)$ and $I(k)$ are the differential transforms of $f(x,u,u')$ and $\int_0^x k(x,s,u(s))ds$, respectively. This recurrence relation is solved and we obtain the series solution of this problem which contains the unknown parameter c_0. Then, the solution obtained is substituted in the second separated boundary condition in (2) to obtain a nonlinear algebraic equation in one unknown parameter which is then solved and the solution is thus obtained.

3. Examples

In this section, we apply the proposed algorithm to two examples that are studies in literature.

Example 2 Consider the Duffing equation [11] and [10]

$$u''(x) + u'(x) + u(x)u'(x) + \int_0^x xsu^2(s)ds = g(x), 0 < x < 1, \tag{11}$$

Where $g(x) = -3x - 3x^2 + \frac{5}{2}x^3 + \frac{2}{3}x^4 - \frac{1}{4}x^5 - \frac{2}{5}x^6 + \frac{1}{6}x^7$, subject to the separated boundary conditions

$$\begin{cases} u(0) - u'(0) = 0, \\ u(1) + u'(1) = 0. \end{cases} \tag{12}$$

From conditions (9) and (12), we assume

$$u(0) = c_0, u'(0) = c_0 \tag{13}$$

Which for $k = 0,1,2,\ldots$ yields the recurrence scheme

$$\begin{cases} U(k+2) = \frac{1}{(k+1)(k+2)}(G(k) - (k+1)U(k+1) - I(k) - N(k), \\ U(0) = c_0, U(1) = c_0, \end{cases} \tag{14}$$

Where $G(k)$ is given by

$$G(k) = -3\delta(k-1) - 3\delta(k-2) + \frac{5}{2}\delta(k-3) + \frac{2}{3}\delta(k-4) - \frac{1}{4}\delta(k-5) - \frac{2}{5}\delta(k-6) + \frac{1}{6}\delta(k-7),$$

$N(k)$ Denotes the differential transform of u^2 which is obtained from the Adomian polynomials for nonnegative integer k as illustrated in Table (1).

Table 1: A_k And $N(k)$ for the power nonlinearity u^2.

k	0	1	2	3
A_k	u_0^2	$2u_0u_1$	$u_1^2 + 2u_0u_2$	$2u_0u_3 + 2u_1u_2$
$N(k)$	$(U(0))^2$	$2U(0)U(1)$	$(U(1))^2 + 2U(0)U(2)$	$2U(0)U(3) + 2U(1)U(2)$

And $I(k)$ is given by

$$I(k) = \begin{cases} 0, & k = 0,1 \\ \frac{1}{k-1}I_1(k-2), & k = 2,3,4,\ldots \end{cases}$$

Where $I_1(k)$ is deduced from the Adomian polynomials of the nonlinear term uu' as illustrated in Table (2).

Table 2: A_k And $I_1(k)$ for the nonlinearity term uu'.

k	0	1	2
A_k	$u_0'u_0$	$u_1'u_0 + u_0'u_1$	$u_2'u_0 + u_1'u_1 + +u_0'u_2$
$I_1(k)$	$U(0)U(1)$	$U(0)U(2) + (U(1))^2$	$3U(0)U(3) + 3U(2)U(1)$

We compute DTM of order $N = 10$ and substitute it in the conditions (12). A nonlinear algebraic equation in the unknown c_0 is obtained for which the solution is given by $c_0 = 1$. This value is substituted in the series solution to get

$$u(x) = 1 + x - x^2 \tag{15}$$

Which is the exact solution of this problem given in [11].

Example 3 Consider the Duffing equation [11] and [10]

$$u''(x) + u'(x) + u(x)(1 + u'(x)) + \int_0^x s^2 u(s)ds = g(x), 0 < x < 1, \tag{16}$$

Where $g(x) = \frac{1}{24(1+e)^2}(204 + 726e^{2x} + 18x + 16x^3 - 9x^4 + 8e^2(42 + 60x + 2x^3 + 3x^4)$

$$+e(540 - 228x + 32x^3 + 15x^4) - 132e^{1+x}(11 + x(x+2)) - 66e^x(11 + x(2x-7))),$$

Subject to the separated boundary conditions

$$\begin{cases} 2u(0) - u'(0) = 0, \\ 3u(1) + u'(1) = 0. \end{cases} \tag{17}$$

From conditions (9), we assume

$$u(0) = c_0, u'(0) = 2c_0 \tag{18}$$

Which for $k = 0,1,2,\ldots$ yields the recurrence scheme

$$\begin{cases} U(k+2) = \frac{1}{(k+1)(k+2)}(G(k) - (k+1)U(k+1) - U(k) - I(k) - N(k), \\ U(0) = c_0, U(1) = c_1 \end{cases} \tag{19}$$

Where $G(k)$ is the differential transform of $g(x)$, which can be computed via the rules of DTM or via Taylor series of $g(x)$, $I(k)$ is given by

$$I(k) = \begin{cases} 0, & k = 0 \\ \frac{1}{k}U(k-3), & k = 1,2,3,\ldots' \end{cases}$$

And $N(k)$ is the differential transform of uu' computed as illustrated in Table (2). We compute DTM of order $N = 50$ and substitute it in the conditions (17). A nonlinear algebraic equation in the unknown parameter c_0 is obtained for which the solution is given by$c_0 = 0.520822$. This value is substituted in the series solution to get

$$u(x) = 0.520822 + 1.04164x - 0.739589x^2 - 0.24653x^3 - 0.0616324x^4 - 0.0123265x^5 \cdots \tag{20}$$

This is Taylor series of

$$u(x) = 2 - \frac{11}{2(1+e^1)}e^x - \frac{3-8e}{2(1+e^1)}x.$$

Which is the exact solution of this problem given in [11].

Table 3: Numerical results of exact solution and absolute error (AE) for Example 2

x	Exact	AE in [11]	AE u_{30} [10]	AE u_{50} [10]	AE DTM u_{50}
0.1	0.617338	$3.36E-5$	$1.12E-12$	$1.1E-16$	0
0.2	0.697493	$3.35E-5$	$2.34E-12$	$2.2E-16$	$2.2E-16$
0.3	0.759565	$3.05E-5$	$3.91E-12$	$2.2E-16$	0
0.4	0.801655	$2.58E-5$	$5.45E-12$	$2.2E-16$	$2.2E-16$
0.5	0.821659	$2.04E-5$	$6.62E-12$	0	0
0.6	0.817256	$1.51E-5$	$7.19E-12$	$4.4E-16$	$4.4E-16$
0.7	0.785877	$1.05E-5$	$7.10E-12$	$2.2E-16$	$4.4E-16$
0.8	0.724687	$6.77E-6$	$6.44E-12$	0	$4.4E-16$
0.9	0.63055	$4.16E-6$	$5.39E-12$	$6.6E-16$	0
1	0.5	$2.71E-6$	$4.16E-12$	$7.7E-16$	0

In Table (3), a numerical comparison between the DTM proposed approach and the schemes in literature where u_n denotes the approximate solution of u with n terms. As shown in table, the results obtained by DTM with only 20 term is better.

4. Conclusion

An approach is presented for using DTM to solve nonlinear Duffing equation that involves forcing terms of both integral and non-integral forms when subjected to separated boundary conditions. This approach reduces the problem to a problem of solving one nonlinear algebraic equation in one unknown parameter. There are some advantages of using this approach in comparison with the ones in literature. No discretizations or linearizations are involved. Also as the method is based on algebraic recurrence relation rather than integral operators, so we can compute as many terms as needed with less computational work. The method presents analytic solution form that coincides with Taylor series of the exact solution. Finally, numerical results for the problem considered using this approach is in good agreement with the exact solution.

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